

Zero-Density Estimates for the Riemann Zeta Function

Stev FIVEZ

Supervisor: Prof. R. Cluckers
KU Leuven, Department of Mathematics,
Algebra Section

Co-supervisor: Prof. Jasson Vindas
UGent, Department of Mathematics:
Analysis, Logic and Discrete Mathematics

Co-supervisor: Dr. Gregory Debruyne
UGent, Department of Mathematics:
Analysis, Logic and Discrete Mathematics

Thesis presented in
fulfillment of the requirements
for the degree of Master of Science
in Mathematics

Academic year 2025-2026

© Copyright by KU Leuven

Without written permission from the supervisors and the authors, it is forbidden to reproduce or adapt in any form or by any means any part of this publication. Requests for obtaining the right to reproduce or utilize parts of this publication should be addressed to KU Leuven, Faculty of Science, Celestijnenlaan 200H - box 2100, 3001 Leuven (Heverlee), Telephone +32 16 32 14 01.

Written permission from the supervisor is also required to use the methods, products, schematics, and programs described in this work for industrial or commercial use, and for submitting this publication in scientific contests.



Copyright Information:

Student paper as part of an academic education and examination.

No correction was made to the paper after examination.

Acknowledgments

As I reflect on the past couple of years, I am struck by the number of incredible people I have met, many of whom supported me throughout my work, including this project. Therefore, this thesis would not be complete without mentioning a few of their names.

First of all, I would like to thank my supervisor, Prof. Raf Cluckers, who enabled me to write a thesis in analytic number theory, a subject not part of the regular curriculum at KU Leuven. Another major thank you goes to my co-supervisors, Prof. Jasson Vindas and Dr. Gregory Debruyne. They introduced me to analytic number theory, a field of mathematics in which I now feel at home, guided me through the project, and patiently answered all my detailed questions. Moreover, Prof. Vindas proposed the project, while Dr. Debruyne suggested the idea that eventually led to a world record in zero-density estimates and also kindly proofread the entire manuscript. The three of them made this project possible, and for that, I am extremely grateful.

Next, I would like to thank the readers, Dr. Loïs Faisant, Dr. Athanasios Kouroupis, and the backup reader, Prof. Arno Kuijlaars, for their interest in my work. A special thanks goes to Dr. Kouroupis for his help in my other academic ventures this year as well.

I would like to express my gratitude to my friends and family for their support as well. In particular, thank you to Abtin and Matin for the weekly workouts and for laughing at my silly videos afterward. Your presence is what made them enjoyable. Another thank you goes to Jargo and Tiberiu for the amusing Friday game sessions and for their continued encouragement, even during my busy periods. I would also like to extend my gratitude to my girlfriend, Aleyna, for her constant love and support.

Finally, I would like to thank my mother. You were always there for me, and I deeply respect what you have been able to achieve on your own.

Introduction

The Riemann zeta function $\zeta(s)$ is one of the most extensively studied functions in pure mathematics. This is for good reason: the theory surrounding this function has many profound implications across analytic number theory and beyond. In particular, it is deeply connected with the prime numbers, something that was first shown by Euler when he proved the Euler product formula

$$\zeta(\sigma) = \prod_p \frac{1}{1 - p^{-\sigma}}, \quad (\sigma > 1)$$

in 1737. However, it was Riemann who extended this function to a complex variable s and thereby discovered the relation of its zeros to the distribution of primes in his revolutionary paper from 1859. At the same time, he formulated his most famous conjecture, which later became known as the Riemann Hypothesis. Namely, he suggested that all non-trivial zeros of the zeta function lie on the so-called critical line $\Re(s) = 1/2$. One of its most important equivalent formulations, illustrating the deep connection to the distribution of primes, amounts to an estimate of the prime counting function $\pi(x)$ with an essentially optimal error term. In particular, if $\pi(x)$ denotes the number of primes not exceeding a real number x , then the equivalent conjecture states that

$$\pi(x) = \int_2^x \frac{1}{\log(t)} dt + O_\varepsilon(x^{1/2+\varepsilon}).$$

The problem is one of the seven Millennium Prize Problems, carrying a prize of \$1,000,000. It has remained unsolved for over 150 years, with few major breakthroughs, and mathematicians still have no clear idea of where to look for a solution. As a result, no attempted proof (of which there have been several) is currently taken seriously, and the problem has been informally described as “one of the hardest ways to win a million dollars.”

Fortunately, over the past century, mathematicians have gained the insight that the full generality of the Riemann Hypothesis is not always required in order to deduce good arithmetic consequences. Notably, in the search for zeros of the zeta function, special attention has been given to zero-density estimates. Rather than improving known zero-free regions, this subject aims to bound the number of zeros $N(\sigma, T)$ in the rectangle defined by $\sigma \leq \Re(s) \leq 1$ and $-T \leq \Im(s) \leq T$ instead. Recently, there has been substantial progress in this area, including a spectacular breakthrough by Guth and Maynard [13]. This has led to the recent initiative known as the Analytic Number Theory Exponent Database (ANTEDB for short) [36], which summarizes the current best results regarding the zeta function.

In this thesis, our aim is to provide a thorough account of the techniques used to prove zero-density estimates. This begins with the description of a popular zero-detection method, which has become standard by now, and proceeds with two examples that together prove the estimate

$$N(\sigma, T) \ll_{\varepsilon} T^{\frac{12}{5}(1-\sigma)+\varepsilon}$$

in the entire interval $1/2 \leq \sigma \leq 1$. Our work then culminates in a new zero-density estimate that improves the current best bounds in a certain interval, effectively breaking a world record. Here, we will exploit recent developments in the theory of exponential sums, namely the discovery of new exponent pairs. A tool that is essential in proving zero-density estimates is the study of moment estimates for the Riemann zeta function on the critical line. These arise naturally in the zero-detection method and, once established, provide a straightforward bound for the contribution of the class-II zeros. This subject has plenty of unsolved problems itself, and we will prove some of the current best results. Most of this is based on Ivić's book [22], which provides an overview of the theory of the zeta function at the time. Nonetheless, his presentation is sometimes concise, and several intermediate steps are omitted. One of the aims of the author is to elaborate on the arguments by providing additional details and filling in the gaps. We conclude with an application of zero-density estimates to the theory of primes in short intervals, where the aforementioned bound will be used. Here, we prove an estimate that is, in a sense, close to what would follow from the Riemann Hypothesis, and significantly stronger than the one implied by the prime number theorem in its current form.

This thesis consists of two parts. Part I is of preliminary nature, building some of the necessary tools when possible, and mentioning useful results otherwise. Chapter 1 introduces the Riemann zeta function and discusses the search for its zeros further. In Chapter 2, we present a Hilbert-type inequality in order to prove two mean value theorems for Dirichlet polynomials, which will be applied multiple times in Chapters 3, 4 and 5. Chapter 3 focuses on exponential sums, a topic with many applications in analytic number theory. In particular, we introduce exponent pairs and mention a result that uses them in the theory of large values of the Riemann zeta function. Exponent pairs will be employed throughout Chapter 5, and this result will allow us to improve the current best zero-density estimates.

Part II is dedicated to Riemann zeta function theory. It starts with a chapter on moment estimates, where we prove an asymptotic formula with an error term for the second and fourth moments. The latter will allow us to establish essentially optimal bounds for all exponents $0 \leq A \leq 4$. We also prove an equivalent estimate for the general case and apply it, together with the theory of large values of the zeta function, to prove the current best known bounds in the interval $4 < A \leq 12$. The main chapter of this thesis is Chapter 5, which is devoted to zero-density estimates. Notably, we establish the current record in the interval $1/2 \leq \sigma \leq 7/10$, before moving on to the proof of our new record estimate. The final chapter serves as an application of zero-density estimates that derives a result on primes in short intervals. This theorem will be compared to the trivial estimate obtained from the prime number theorem with its best known error term, and a consequence for gaps between primes will be presented. We end with a reflection and a discussion of possible improvements to our new zero-density estimate.

List of symbols

s, z, w	Variable complex numbers.
$\Re(s)$	The real part of s .
$\Im(s)$	The imaginary part of s .
$s = \sigma + it$	We often implicitly assume that σ denotes the real part of s and t the imaginary part.
$\mathbb{C}_\theta$	The half-plane $\{z \in \mathbb{C} : \Re(z) > \theta\}$, where $\theta \in \mathbb{R}$.
$\mathbb{N}$	The set of positive integers, i.e. $\mathbb{Z}_{>0}$.
$\zeta(s)$	The Riemann zeta function evaluated in $s \in \mathbb{C} \setminus \{1\}$.
$\Gamma(s)$	The Gamma function evaluated in $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.
$\chi(s)$	The function $\chi(s) := 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s)$ evaluated in $s \in \mathbb{C} \setminus (2\mathbb{Z}_{\geq 0} + 1)$.
$\rho = \beta + i\gamma$	A non-trivial zero of the Riemann zeta function with real part β and imaginary part γ .
ε	A positive parameter that can attain any sufficiently small value.
$\sum_{n \in X}, \sum_{m \in X}$	A sum over all integers $n \geq 1$ (or $m \geq 1$ in the second sum) contained in the set X . If the sum is empty, it equals zero.
$\sum_{p \in X}$	A sum over all prime numbers p contained in the set X .
$\sum_{\rho \in X}$	A sum over all non-trivial zeros ρ of the Riemann zeta function contained in the set X .
$\sum_{p^k \in X}$	A sum over all prime powers p^k ($k \geq 1$) contained in the set X .
$\prod_{p \in X}$	A product over all prime numbers p contained in the set X . If the product is empty, it equals 1.
$\{x\}$	The fractional part of $x \in \mathbb{R}$, i.e. $\{x\} := x - \lfloor x \rfloor$.

$\lfloor x \rfloor$	The largest integer not exceeding $x \in \mathbb{R}$, i.e. $\lfloor x \rfloor := \max(\mathbb{Z} \cap (-\infty, x])$.
$\log(x)$	The natural logarithmic function evaluated in $x \in (0, \infty)$.
$1_A(x)$	The indicator function of a subset $A \subseteq X$, defined as $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in X \setminus A$.
$\mu(\sigma)$	The growth exponent function for the Riemann zeta function, defined as the infimum of all $\theta \in \mathbb{R}$ such that $\zeta(\sigma + it) \ll_{\theta} t ^{\theta}$ for sufficiently large $ t $.
$N(T)$	The number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 \leq \gamma \leq T$, counted with multiplicity.
$N(\sigma, T)$	The number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\sigma \leq \beta \leq 1$ and $-T \leq \gamma \leq T$, counted with multiplicity.
$f(x) = O(g(x))$	Landau O -notation. We say that this equality holds uniformly over a set X if there exists some $C > 0$ such that $ f(x) \leq C g(x) $ for all $x \in X$. If no set is specified, we assume that $X \subseteq \mathbb{R}$ and the inequality holds for sufficiently large x , meaning that there exists some $x_0 \in \mathbb{R}$ such that the inequality holds for all $x \in X$ with $x \geq x_0$.
$f(x) = O_{a,b}(g(x))$	The same as the above notation, but where the constant C may depend on the value of the parameters a or b . If we assume a parameter to be fixed, we usually omit it from the subscript.
$f(x) \ll_{a,b} g(x)$	This means the same as $f(x) = O_{a,b}(g(x))$.
$f(x) \asymp_{a,b} g(x)$	This means that both $f(x) \ll_{a,b} g(x)$ and $f(x) \gg_{a,b} g(x)$ hold.
$f(x) = o(g(x))$	Landau o -notation, meaning $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$, assuming that $g(x) \neq 0$ for sufficiently large x .
$f(x) \sim g(x)$	Asymptotic equivalence of functions as $x \rightarrow \infty$, meaning that $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$, assuming that $g(x) \neq 0$ for sufficiently large x .
$\pi(x)$	The prime counting function, defined as the number of primes not exceeding the real number x .
$\vartheta(x)$	The function defined by $\vartheta(x) := \sum_{p \leq x} \log(p)$.
$\psi(x)$	The Chebyshev function, defined as $\psi(x) := \sum_{p^k \leq x} \log(p)$.
$\ u\ $	The norm of an element u in a complex vector space.
$\langle u, v \rangle$	An inner product in a complex vector space, evaluated in the pair of elements (u, v) .

(κ, λ)	An exponent pair.
$e(x)$	The function $e(x) := e^{2\pi i x}$ evaluated in $x \in \mathbb{R}$.
$\widehat{f}(t)$	The Fourier transform of f evaluated in $t \in \mathbb{R}$.
$f^n(x)$	This denotes $f(x)$ raised to the power n , where n is an integer.
$d(n)$	The divisor function, defined as the number of positive divisors of an integer $n \geq 1$.
$d_k(n)$	The k -fold divisor function, defined as the number of ways we can write a positive integer n as $n = n_1 \dots n_k$, where $(n_1, \dots, n_k) \in \mathbb{N}^k$.
$\mu(n)$	The Möbius function evaluated in an integer $n \geq 1$, defined as $\mu(n) = (-1)^k$ if $n = p_1 \dots p_k$ is the prime decomposition of n , and zero otherwise.
γ	The Euler-Mascheroni constant.
$M(A)$	The infimum of all $M \in \mathbb{R}$ such that $\int_0^T \zeta(1/2 + it) ^A dt \ll_M T^M$, where $A \geq 0$ is fixed.

Table of contents

Acknowledgments	iii
Introduction	v
List of symbols	vii
I Tools from Analytic Number Theory	1
1. The Riemann zeta function	3
1.1. Definition	3
1.2. The functional equation	4
1.3. The zeros	5
1.4. Some estimates	8
2. Mean value theorems for Dirichlet polynomials	13
2.1. A Hilbert-type inequality	14
2.2. The mean value theorem for Dirichlet polynomials (integral form)	15
2.3. The mean value theorem for Dirichlet polynomials (discrete form)	21
3. Exponential sums	25
3.1. Preliminary lemmas	25
3.2. Exponent pairs	28
3.3. Application to large values of the Riemann zeta function	31
II Theory of the Riemann Zeta Function	37
4. Moment estimates on the critical line	39
4.1. The second moment	39
4.2. The fourth moment	41
4.3. Even power moments	50
4.4. Real power moments	51
4.4.1. The value of $M(A)$ for exponents $0 \leq A \leq 4$	52
4.4.2. An equivalent estimate	52
4.4.3. The current record for exponents $4 \leq A \leq 12$	54
4.4.4. The conjectured value of $M(A)$ and its relation to the Lindelöf hypothesis	55

5. Zero-density estimates	57
5.1. The zero-detection method, Part I	58
5.2. A sharp estimate for $1/2 \leq \sigma \leq 3/4$	61
5.3. The zero-detection method, Part II	63
5.3.1. The class-I zeros	63
5.3.2. The class-II zeros	65
5.4. A crude estimate for $3/4 \leq \sigma \leq 1$	65
5.5. Breaking a world record around $\sigma = 9/10$	68
5.5.1. Ivić's method to bound the contribution of the class-I zeros	69
5.5.2. Optimizing the exponent pair	72
5.5.3. Final calculations	74
6. Applications to primes in short intervals	75
6.1. An estimate via the prime number theorem	76
6.2. A stronger result using zero-density estimates	76
6.3. A consequence for gaps between primes	80
Conclusion	83
Appendix	85
References	87

Part I

Tools from Analytic Number Theory

Chapter 1

The Riemann zeta function

The Riemann zeta function plays a central role in analytic number theory. While initially introduced by Euler, it was Riemann who extended the function to a complex variable and connected it to the distribution of primes in his revolutionary paper [32]. In that work, he also introduced the Riemann Hypothesis, one of the most important unsolved problems in pure mathematics, with profound implications across many areas of mathematics.

In this chapter, we introduce a few basic concepts and results regarding the zeta function that are necessary for this thesis, and motivate the search for its zeros by explaining the relation to the distribution of primes. However, the literature surrounding this function is vast, and we refer the interested reader to [22, 38].

1.1. Definition

Initially, the Riemann zeta function is defined as the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1.1)$$

which converges in $\mathbb{C}_1$, where we denote $\mathbb{C}_\theta = \{z \in \mathbb{C} : \Re(z) > \theta\}$ for any $\theta \in \mathbb{R}$. However, it can be easily seen that the function extends meromorphically to the right half-plane $\mathbb{C}_0$. One approach to verify this uses a different Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s},$$

which converges in $\mathbb{C}_0$. Summing and subtracting two times the even terms gives for $\Re(s) > 1$:

$$G(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = (1 - 2^{1-s})\zeta(s).$$

This implies that the zeta function has a meromorphic continuation to $\mathbb{C}_0$, the only possible poles being located at $1 + 2\pi in / \log(2)$ for $n \in \mathbb{Z}$. Another way to come to this conclusion is by using Euler's summation formula to get

$$\zeta(s) = \frac{s}{s-1} - \int_1^{\infty} \{x\} x^{-s-1} dx$$

if $\Re(s) > 1$. Here, we denote $\{x\} = x - \lfloor x \rfloor$ as the fractional part of a real number x . The right-hand side delivers the meromorphic continuation of $\zeta(s)$ to $\mathbb{C}_0$ and has exactly one pole, namely at $s = 1$ with residue 1.

A more sophisticated argument (see e.g. [37, Part II, Thm. 3.1]) establishes that $\zeta(s)$ can be meromorphically extended even further to the entire complex plane. This extension does not produce any additional singularities besides the simple pole at $s = 1$.

1.2. The functional equation

The following famous functional equation relates the evaluation of the zeta function in two values mirrored around the point $s = 1/2$. It will help us locate zeros in the left half-plane in the next section, as well as find a symmetry in the location of zeros in the critical strip. Moreover, it will also be used to estimate the fourth moment of the zeta function on the critical line $\Re(s) = 1/2$ in Chapter 4.

Theorem 1.1 (see e.g. [37, Part II, Thm. 3.3]). For every complex number s that is no positive odd integer, we have

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s) \zeta(1-s). \quad (1.2)$$

Here we introduced the Gamma function $\Gamma(s)$. In the next section, we will see that $\zeta(s)$ has a zero at every negative even integer, making (1.2) valid for all $s \in \mathbb{C} \setminus \{1\}$.

We usually denote

$$\chi(s) := 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s). \quad (1.3)$$

Since poles cancel with zeros at positive even integers, this function is meromorphic on the whole complex plane, with (simple) poles only in the positive odd integers. The functional equation then becomes

$$\zeta(s) = \chi(s) \zeta(1-s) \quad (1.4)$$

for $s \in \mathbb{C} \setminus \{1\}$.

In the proof of the fourth moment estimate in Chapter 4, we will substitute the functional equation into an integral over a vertical line in the complex plane. In order to then change the path of integration to a different vertical line without creating additional terms due to the horizontal parts, it will be useful to have an estimate of $\chi(s)$ in vertical strips. Such a result will be easily achievable once we have a similar estimate for the Gamma function. Therefore, we record [37, Part II, Cor. 0.13] in the next theorem. This result is a consequence of the complex Stirling formula and will itself be applied repeatedly in Chapters 4 and 5.

Theorem 1.2. Let $\sigma_1 < \sigma_2$ be real numbers. Then, uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \rightarrow \infty$, we have¹

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|} \left(1 + O\left(\frac{1}{t}\right)\right). \quad (1.5)$$

Corollary 1.3. Let $\sigma_1 < \sigma_2$ be fixed real numbers. Then, uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \geq 2$,²

$$|\chi(s)| \asymp (2\pi)^\sigma |t|^{\frac{1}{2}-\sigma}.$$

Proof. We use the previous theorem and the definition of the sine function for $|t|$ big enough:

$$|\chi(s)| \leq 2^\sigma \pi^{\sigma-1} \frac{e^{-\frac{\pi}{2}|t|} + e^{\frac{\pi}{2}|t|}}{2} \sqrt{2\pi}|t|^{\frac{1}{2}-\sigma} e^{-\frac{\pi}{2}|t|} \left(1 + O\left(\frac{1}{t}\right)\right) \ll (2\pi)^\sigma |t|^{\frac{1}{2}-\sigma},$$

and

$$|\chi(s)| \geq 2^\sigma \pi^{\sigma-1} \frac{e^{\frac{\pi}{2}|t|} - e^{-\frac{\pi}{2}|t|}}{2} \sqrt{2\pi}|t|^{\frac{1}{2}-\sigma} e^{-\frac{\pi}{2}|t|} \left(1 + O\left(\frac{1}{t}\right)\right) \gg (2\pi)^\sigma |t|^{\frac{1}{2}-\sigma}.$$

We can extend these estimates to $|t| \geq 2$ using continuity and the fact that $\chi(s) \neq 0$ in this domain. $\square$

1.3. The zeros

One of the most important subjects of study regarding the Riemann zeta function is the location of its zeros. The zeros in the strip $0 < \Re(s) < 1$, which is known as the critical strip, are of particular importance due to their connection to the distribution of primes.

We start off by showing that the zeta function does not have zeros in the closed half-plane defined by $\Re(s) \geq 1$. The functional equation (1.2) then provides a full description of the zeros in the left half-plane $\Re(s) \leq 0$. Namely, the only factor in the functional equation that can create zeros for $\sigma \leq 0$ is $\sin(\pi s/2)$, which does so if and only if s is a non-positive even integer. However, in $s = 0$, this zero cancels with the pole of $\zeta(1-s)$. Hence, we conclude that the only zeros in the closed left half-plane are the negative even integers. We call these the trivial zeros.

If $\sigma > 1$, then the Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely. Since the Dirichlet

¹We usually write $s = \sigma + it$ to denote the real and imaginary part of a complex number s .

²In the proof, we adopt Vinogradov's symbol $\ll$, writing $f(x) \ll g(x)$ to mean $f(x) = O(g(x))$, i.e. $|f(x)| \leq C|g(x)|$ for some absolute constant $C \geq 0$. We further denote $f(x) \asymp g(x)$ if $f(x) \ll g(x)$ and $f(x) \gg g(x)$.

coefficients are given by the completely multiplicative sequence $a(n) = 1$, we may write the function as the Euler product

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}. \quad (1.6)$$

Moreover, the infinite product converges absolutely, and none of the factors are zero, so $\zeta(s)$ is non-zero if $\sigma > 1$.

The argument showing that the zeta function has no zeros on the line with real part 1 is more difficult, but still very manageable. We present a very standard proof, starting with a lemma.

Lemma 1.4 (The 3-4-1 inequality for $\zeta(s)$). For $\sigma > 1$ and real t , we have

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$$

Proof. We start from the Euler product (1.6). Since $|1/p^s| < 1$ if $\sigma > 1$, we can use the usual complex logarithm defined on $\mathbb{C} \setminus \mathbb{R}^-$ as follows:

$$\log |\zeta(s)| = - \sum_p \log \left| 1 - \frac{1}{p^s} \right| = - \sum_p \Re \left(\log \left(1 - \frac{1}{p^s} \right) \right).$$

Applying the exponential function on both sides and substituting the Taylor series of $\log(1 - s)$ gives

$$|\zeta(s)| = \exp \left(\sum_p \Re \left(\sum_{n=1}^{\infty} \frac{1}{np^{ns}} \right) \right) = \exp \left(\sum_p \sum_{n=1}^{\infty} \frac{\cos(nt \log(p))}{np^{n\sigma}} \right).$$

Hence, if $\sigma > 1$, it follows that

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| = \exp \left(\sum_p \sum_{n=1}^{\infty} \frac{3 + 4 \cos(nt \log(p)) + \cos(2nt \log(p))}{np^{n\sigma}} \right).$$

We finish the proof by noting that

$$3 + 4 \cos(\theta) + \cos(2\theta) = 2(1 + \cos(\theta))^2 \geq 0$$

for any real θ . □

Now suppose that $\zeta(1 + it) = 0$ for some fixed real $t \neq 0$ (if $t = 0$, we have a pole). The Taylor series in this point provides the estimate

$$\zeta(\sigma + it) = O(\sigma - 1), \quad \sigma \rightarrow 1.$$

Moreover, we have $\zeta(\sigma + 2it) = O(1)$, $\sigma \rightarrow 1$ by the continuity of the zeta function and $\zeta(\sigma) = O(\frac{1}{\sigma-1})$, $\sigma \rightarrow 1$ because of the pole at the point 1. Combining this with the previous lemma gives

$$1 \leq |\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| = O(\sigma - 1), \quad \sigma \rightarrow 1^+,$$

which is a contradiction. Hence, we can conclude that the zeta function does not have zeros on the line with real part 1.

We are therefore left with determining the non-trivial zeros, namely the zeros inside the critical strip. Observing the symmetry of the distribution of zeros in this strip, we can limit our study even further to $1/2 \leq \Re(s) < 1$. Indeed, the functional equation (1.2) tells us that a complex number s with $0 < \sigma < 1$ is a zero of the zeta function if and only if $1 - s$ is a zero. Moreover, we have $\overline{\zeta(\bar{s})} = \zeta(s)$ since $\zeta(s)$ is analytic on a region symmetric with respect to the real axis. Hence, a complex number s inside the critical strip is a zero if and only if $\bar{s}$ is. We conclude that the location of the zeros inside the critical strip is symmetric with respect to both the real line and the line with real part $1/2$.

This is, however, where things start to become quite difficult. In fact, determining the remaining zeros has been an open problem for more than 160 years. The famous Riemann Hypothesis conjectures that they are all located on the so-called critical line $\Re(s) = 1/2$. A solution to this problem would have profound implications in analytic number theory, notably for the distribution of primes. In particular, it would render all results in the theory of zero-density estimates (and therefore Chapter 5) redundant. The conjecture is part of Hilbert's 23 problems for the 20th century and one of the seven Millennium Prize Problems for the 21st century.

At present, we are not even close to proving a zero-free region of the form $\Re(s) > 1 - \varepsilon$ for some $\varepsilon > 0$. The current best zero-free region, according to the ANTEDB (the Analytic Number Theory Exponent Database) [36, Chapter 13], is recorded in the next theorem. The result is obtained via the methods of Korobov [25] and Vinogradov [41], and a full proof is presented in [22, Chapter 6]. This zero-free region is, along with a specific type of zero-density estimate, the main ingredient in our proof that estimates the number of primes in short intervals in Chapter 6.

Theorem 1.5 (Korobov-Vinogradov zero-free region). There exist an absolute constant $A > 0$ such that $\zeta(s)$ has no zeros in the region determined by

$$\sigma \geq 1 - \frac{A}{(\log |t|)^{2/3} (\log \log |t|)^{1/3}}, \quad |t| \geq 3.$$

Another reason to seek an improved zero-free region is that it leads to a better error term when estimating the prime counting function $\pi(x)$. This can be seen, for instance, through a standard argument such as [37, Thm. 4.1]. Here, the idea is to apply a Perron inversion formula on the Dirichlet series $\zeta'(s)/\zeta(s)$ of the von Mangoldt function and to use the residue theorem to move the path of integration left of the line $\Re(s) = 1$. Here, we have to avoid zeros of the zeta function to prevent the appearance of additional terms. The further to the left we go, the better the estimate becomes. In particular, a zero-free region of the form $\mathbb{C}_{1-\varepsilon}$ would already significantly improve the current best error term (which is the same as (6.3) below). The following theorem illustrates this deep connection between the zeros of the zeta function and the distribution of primes.

Theorem 1.6 ([22, Thm. 12.3]). Fix $1/2 \leq \theta \leq 1$. Then

$$\pi(x) = \int_2^x \frac{1}{\log(t)} dt + O(x^\theta \log(x))$$

if and only if the half-plane $\mathbb{C}_\theta$ contains no zeros of $\zeta(s)$.

It is known that there are infinitely many non-trivial zeros. This follows, for example, from the following famous estimate, whose proof relies on the argument principle (see e.g. [37, Part II, Theorem 3.19]). We usually denote a non-trivial zero of the zeta function by $\rho = \beta + i\gamma$. Then, let $N(T)$ be the number of non-trivial zeros $\rho = \beta + i\gamma$ with $0 \leq \gamma \leq T$, counted with multiplicity.

Theorem 1.7 (Riemann-von Mangoldt formula). For $T \geq 2$, we have

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log(T)). \quad (1.7)$$

In particular, by the symmetry of the non-trivial zeros with respect to the critical line, we know that there are zeros in the strip $1/2 \leq \Re(s) < 1$. Therefore, in Theorem 1.6, we have $\Theta \geq 1/2$.

If we now fix $\sigma > 1/2$ and only consider the zeros $\rho = \beta + i\gamma$ with $\gamma \geq \sigma$ instead, there exist much smaller estimates than Theorem 1.7. This is not unexpected, since the Riemann Hypothesis predicts that there are no zeros here at all. These results are known as zero-density estimates for the Riemann zeta function, which will be further discussed in Chapter 5.

1.4. Some estimates

In this section, we summarize a few estimates regarding the zeta function that will prove to be useful later in this thesis. The first result aims to estimate $\zeta(s)$ in the critical strip by truncating its partial sums. This will allow us to easily prove the second moment estimate of the zeta function on the critical line in Chapter 4.

Proposition 1.8 ([22, Theorem 1.8]). Fix $0 < \delta < 1$. Then, uniformly for $\sigma > 0$, $t \in \mathbb{R}$ and $x \in [\frac{|t|}{2\pi\delta}, \infty) \setminus \{0\}$, we have

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}). \quad (1.8)$$

Proof. For $\sigma > 1$ and $y > 0$, Euler's summation formula gives

$$\zeta(s) - \sum_{n \leq y} \frac{1}{n^s} = \sum_{n > y} \frac{1}{n^s} = -\frac{y^{1-s}}{1-s} - s \int_y^\infty \frac{\{t\}}{t^{s+1}} dt + \frac{\{y\}}{y^s}. \quad (1.9)$$

By analytic continuation, this equation extends to $s \in \mathbb{C}_0 \setminus \{1\}$.

The idea is to now estimate the sum

$$\sum_{x < n \leq y} \frac{1}{n^s}$$

using partial summation and the theory of exponential sums (see Chapter 3), substitute this into (1.9) and let $y \rightarrow \infty$. Define the sequence $a(n)$ by $a(n) = 0$ if $n \leq x$ and $a(n) = n^{-it}$ if $n > x$. For $\sigma > 0$ and $y > x > 0$, the summation by parts formula gives

$$\sum_{x < n \leq y} \frac{1}{n^s} = \sum_{n \leq y} a(n) n^{-\sigma} = \sigma \int_x^y u^{-\sigma-1} \left(\sum_{x < n \leq u} n^{-it} \right) du + y^{-\sigma} \left(\sum_{x < n \leq y} n^{-it} \right). \quad (1.10)$$

We now use Lemma 3.1, proven below, to estimate the exponential sums. In this lemma, we choose the function $f(u) = -(2\pi)^{-1}t \log(u)$. Provided that $x \geq \frac{|t|}{2\pi\delta}$, we have

$$\sum_{x < n \leq u} n^{-it} = \int_x^u v^{-it} dv + O(1) = \frac{u^{1-it} - x^{1-it}}{1-it} + O(1).$$

Plugging this into (1.10) gives

$$\begin{aligned} \sum_{x < n \leq y} \frac{1}{n^s} &= \sigma \int_x^y \frac{u^{-s} - u^{-\sigma-1} x^{1-it}}{1-it} du + O(x^{-\sigma}) + \frac{y^{1-s}}{1-it} + O(xy^{-\sigma}) \\ &= \frac{1}{1-it} \left(\frac{\sigma}{1-s} + 1 \right) (y^{1-s} - x^{1-s}) + \frac{y^{-\sigma} x^{1-it}}{1-it} + O(x^{-\sigma}) + O(xy^{-\sigma}) \\ &= \frac{y^{1-s}}{1-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) + O(xy^{-\sigma}). \end{aligned}$$

Inserting this into (1.9) gives

$$\begin{aligned} \zeta(s) - \sum_{n \leq x} \frac{1}{n^s} &= \sum_{x < n \leq y} \frac{1}{n^s} - \frac{y^{1-s}}{1-s} - s \int_y^\infty \frac{\{t\}}{t^{s+1}} dt + \frac{\{y\}}{y^s} \\ &= -\frac{x^{1-s}}{1-s} + O(x^{-\sigma}) + O(xy^{-\sigma}) - s \int_y^\infty \frac{\{t\}}{t^{s+1}} dt + \frac{\{y\}}{y^s}. \end{aligned}$$

Letting y tend to infinity completes the proof. $\square$

We continue with an elementary estimate of the growth pattern of the zeta function on vertical lines in the complex plane. Similarly to the estimate of $\chi(s)$ in Corollary 1.3, this will be useful for the fourth moment estimate. A crude estimate can already be obtained from the previous proposition, and will prove polynomial growth in all vertical strips, which is good enough for most applications in this text.

First note that, for all $\varepsilon > 0$, $\zeta(s)$ is bounded in $\mathbb{C}_{1+\varepsilon}$ since its defining Dirichlet series (1.1)

converges absolutely in the closure of this half-plane. Next, we can apply the previous proposition for $\delta = 1/2$, $1/2 \leq \sigma \leq 1$, and $x = |t| \geq 2$ to get

$$\zeta(\sigma + it) \ll \sum_{n \leq |t|} \frac{1}{n^\sigma} - \frac{|t|^{1-\sigma}}{|t|} + O(|t|^{-\sigma}) \ll \begin{cases} \log |t| & \text{if } \sigma = 1, \\ \frac{1}{1-\sigma} |t|^{1-\sigma} & \text{else.} \end{cases}$$

In order to make the constant in the estimate independent of σ , we can replace the exponent σ in the sum by $1/2$ to gain $\zeta(\sigma + it) \ll |t|^{1/2}$ uniformly in $\mathbb{C}_{1/2}$. Finally, we obtain a bound for $\sigma < 1/2$ from the functional equation (1.4) combined with the estimate of $\chi(s)$ in Corollary 1.3. In summary, if we fix $\sigma_0 < 0$, we get

$$\zeta(\sigma + it) \ll \begin{cases} |t|^{1/2} & \text{if } \sigma \geq 1/2, \\ |t|^{1-\sigma} & \text{if } \sigma < 1/2, \end{cases}$$

uniformly for $\sigma \geq \sigma_0$ and $|t| \geq 2$.

We now present a more refined but still elementary bound. The estimates for $\sigma \geq 1$ and $\sigma_0 \leq \sigma \leq 0$ can be achieved similarly as before, while the Phragmén-Lindelöf principle was used to achieve the estimate in $0 < \sigma < 1$.

Theorem 1.9 ([22, Thm. 1.9]). Fix $\varepsilon > 0$. There exists some $t_0 \geq 2$ such that, uniformly in $\sigma \in \mathbb{R}$ and $|t| \geq t_0$,

$$\zeta(\sigma + it) \ll \begin{cases} 1 & \text{if } \sigma \geq 1 + \varepsilon, \\ \log |t| & \text{if } 1 \leq \sigma < 1 + \varepsilon, \\ |t|^{\frac{1-\sigma}{2}} \log |t| & \text{if } 0 \leq \sigma < 1, \\ |t|^{\frac{1}{2}-\sigma} \log |t| & \text{if } \sigma < 0. \end{cases}$$

In order to estimate the Riemann zeta function on vertical strips, one often introduces the growth exponent function $\mu(\sigma)$, defined as the infimum of all real numbers θ such that

$$\zeta(\sigma + it) \ll_\theta |t|^\theta, \quad t \rightarrow \infty.$$

Theorem 1.9 ensures that $\mu(\sigma) < \infty$ holds everywhere and shows that $\mu(\sigma) \leq 0$ for $\sigma \geq 1$. Moreover, if σ is sufficiently large, say $\sigma \geq \sigma_0$, then

$$|\zeta(s)| \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^\sigma} > 0.$$

Hence, $\mu(\sigma) = 0$ for $\sigma \geq \sigma_0$. In addition, it is well known (see e.g. [37, Part II, Section 1.6]) that μ is a convex, non-increasing function, so $\mu(\sigma) = 0$ if $\sigma \geq 1$. The functional equation (1.4), together with the estimate of $\chi(s)$ in Corollary 1.3, then shows that $\mu(\sigma) = 1/2 - \sigma$ for $\sigma \leq 0$. The symmetry imposed by the functional equation with respect to the line $\Re(s) = 1/2$ suggests that the value $\mu(1/2)$ plays an important role. Therefore, it has been studied extensively, and its upper bound has been lowered numerous times in the

past century, see [36, Table 6.1]. The current record is given in the theorem below. In particular, the fact that $\mu(1/2) < 1/6$ will be exploited a few times throughout this thesis.

Theorem 1.10 (Bourgain [7]).

$$\mu(1/2) \leq 13/84.$$

It is conjectured that $\mu(1/2) = 0$, a statement known as Lindelöf's hypothesis. In that case, the value of $\mu(\sigma)$ is determined on the whole real line by the convexity of μ , namely

$$\mu(\sigma) = \begin{cases} 0 & \text{if } \sigma \geq 1/2, \\ \frac{1}{2} - \sigma & \text{else.} \end{cases}$$

This conjecture is itself implied by the Riemann Hypothesis, see e.g. [22, Section 1.9].

While establishing zero-density estimates, we will make use of moment estimates on the critical line to easily bound the total number of so-called class-II zeros. However, we will not deal with moments directly, but instead need an equivalent estimate for sums of powers of the zeta function evaluated in well-spaced values, established in Subsection 4.4.2. To prove this equivalence, it will be useful to estimate one value of the zeta function to a power by a small moment integral of the same power. Therefore, we mention the following lemma due to Ivić [22, Lem. 7.1]. However, he only stated this result for integer powers, so we generalize it for any real exponent $A \geq 1$ as a corollary, but sacrifice an extra logarithmic factor in the process.

Lemma 1.11. Let $k \geq 1$ be a fixed integer. Then, uniformly for $T \geq 2$ and $T/2 \leq t \leq 2T$,

$$|\zeta(1/2 + it)|^k \ll \log(T) \left(1 + \int_{-\log^2(T)}^{\log^2(T)} |\zeta(1/2 + it + iv)|^k e^{-|v|} dv \right).$$

Corollary 1.12. Let $A \geq 1$ be a fixed real number. Then, uniformly for $T \geq 2$ and $T/2 \leq t \leq 2T$,

$$|\zeta(1/2 + it)|^A \ll \log^4(T) \left(1 + \int_{-\log^2(T)}^{\log^2(T)} |\zeta(1/2 + it + iv)|^A e^{-|v|} dv \right).$$

Proof. We apply the lemma to the positive integer k for which $k \leq A < k + 1$ and use Jensen's inequality, where we divide by $2 \log^2(T)$ to gain a probability measure in the integral. This yields

$$\begin{aligned} & |\zeta(1/2 + it)|^A \\ & \ll \log^{\frac{A}{k}}(T) \left(1 + \int_{-\log^2(T)}^{\log^2(T)} |\zeta(1/2 + it + iv)|^k e^{-|v|} dv \right)^{\frac{A}{k}} \end{aligned}$$

$$\begin{aligned}
&\leq \log^2(T) \cdot 2^{\frac{A}{k}} \left(1 + \left(2 \log^2(T) \int_{-\log^2(T)}^{\log^2(T)} |\zeta(1/2 + it + iv)|^k e^{-|v|} \frac{dv}{2 \log^2(T)} \right)^{\frac{A}{k}} \right) \\
&\leq 4 \log^2(T) \left(1 + (2 \log^2(T))^{\frac{A}{k}-1} \int_{-\log^2(T)}^{\log^2(T)} |\zeta(1/2 + it + iv)|^A e^{-\frac{A}{k}|v|} dv \right) \\
&\ll \log^4(T) \left(1 + \int_{-\log^2(T)}^{\log^2(T)} |\zeta(1/2 + it + iv)|^A e^{-|v|} dv \right). \quad \square
\end{aligned}$$

Chapter 2

Mean value theorems for Dirichlet polynomials

While proving zero-density estimates, the contribution of the class-I zeros admits a natural bound involving Dirichlet polynomials. This is not surprising, as these functions pop up frequently in analytic number theory, including in this thesis. Namely, they serve as finite approximations of Dirichlet series, which include the zeta function and many L -functions, and are therefore easier to handle. Consequently, obtaining good bounds often reduces to estimating Dirichlet polynomials.

Recall that a Dirichlet polynomial is a function of the type

$$P(s) = \sum_{n=1}^N \frac{a_n}{n^s},$$

where $a_1, \dots, a_N$ are complex numbers and $s \in \mathbb{C}$ is a complex variable. This chapter is concerned with estimating the second moment

$$\int_0^T |P(it)|^2 dt,$$

and the related sum

$$\sum_{r \leq R} |P(it_r)|^2$$

over well-spaced values $t_1, \dots, t_R$. Both are known as the mean value theorem for Dirichlet polynomials and they are respectively referred to as the integral and discrete variants.

The integral version is essential in proving moment estimates for the Riemann zeta function on the critical line. In fact, it will provide the main term for both the second and fourth moments. The discrete form will be useful in zero-density estimates to bound a number of well-spaced class-I zeros. In the interval $1/2 \leq \sigma \leq 3/4$, this turns out to give sharper results than the Halász-Montgomery inequality, which is used in the remaining interval $3/4 \leq \sigma \leq 1$. The discrete form is also used to obtain a result regarding large values of the zeta function in Section 3.3.

The proof of the integral form of the mean value theorem will crucially rely on a certain Hilbert-type inequality. Therefore, we first establish this result.

2.1. A Hilbert-type inequality

We begin with a proof of two inequalities, which are both known as a type of Hilbert inequality. Each inequality will be used in the proof of the integral variant of the mean value theorem for Dirichlet polynomials, while the first inequality is also used to prove the second one. A proof of both results can be found in [22, Section 5.2], but for the second inequality, we use a different argument that gives a better constant. Moreover, it allows for an easier generalization once a more general inequality than (2.1) is available.

Proposition 2.1. Suppose that $a_1, \dots, a_N, b_1, \dots, b_N$ are complex numbers and $q_1, \dots, q_N$ distinct integers, then the following two inequalities hold:

$$\left| \sum_{n \neq m \leq N} \frac{a_n \overline{a_m}}{q_n - q_m} \right| \leq \pi \sum_{n \leq N} |a_n|^2, \quad (2.1)$$

$$\left| \sum_{n \neq m \leq N} \frac{a_n \overline{b_m}}{q_n - q_m} \right| \leq 2\pi \left(\sum_{n \leq N} |a_n|^2 \right)^{1/2} \left(\sum_{n \leq N} |b_n|^2 \right)^{1/2}. \quad (2.2)$$

Proof. For (2.1), we shorten notation by introducing

$$E = \sum_{n \neq m \leq N} \frac{a_n \overline{a_m}}{q_n - q_m}.$$

Remark that $\overline{E} = -E$, since the roles of n and m swap in the numerator by taking a complex conjugation and in the denominator by taking the additive inverse. Hence, E is purely imaginary. In case $E/i \geq 0$, we make the following calculation:

$$\begin{aligned} 0 &\leq \int_0^1 \int_0^y \left| \sum_{n \leq N} a_n e^{2\pi i q_n x} \right|^2 dx dy \\ &= \int_0^1 \int_0^y \left(\sum_{n \leq N} a_n e^{2\pi i q_n x} \right) \left(\sum_{m \leq N} \overline{a_m e^{2\pi i q_m x}} \right) dx dy \\ &= \sum_{n \neq m \leq N} a_n \overline{a_m} \int_0^1 \int_0^y e^{2\pi i (q_n - q_m)x} dx dy + \sum_{n \leq N} a_n \overline{a_n} \int_0^1 \int_0^y 1 dx dy \\ &= -\frac{1}{2\pi i} E + \frac{1}{2} \sum_{n \leq N} |a_n|^2, \end{aligned} \quad (2.3)$$

and therefore

$$|E| = \frac{E}{i} \leq \pi \sum_{n \leq N} |a_n|^2.$$

In case $E/i < 0$, we perform the same computation, but with $\left| \sum_{n \leq N} a_n e^{2\pi i q_n x} \right|^2$ replaced by $\left| \sum_{n \leq N} a_n e^{-2\pi i q_n x} \right|^2$ in the beginning. This gives the same result upon noting that $|E| = -\overline{E}/i$. Hence, the first inequality is proven.

The second inequality follows from some linear algebra in the Hilbert space $\mathbb{C}^N$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$. First, define the matrix T by setting $T_{ij} = 1/(q_j - q_i)$ if $i \neq j$ and $T_{ii} = 0$, so that

$$\sum_{n \neq m \leq N} \frac{a_n \overline{b_m}}{q_n - q_m} = \langle Ta, b \rangle.$$

Note that the transpose $T^t = -\overline{T}$, i.e. T is a Skew-symmetric matrix. Therefore, the map $(a, b) \mapsto \langle Ta, b \rangle$ is a Skew-Hermitian form, meaning that $(a, b) \mapsto i\langle Ta, b \rangle$ is a Hermitian form. Therefore, we may employ the polarization formula, where we denote $\|x\|_T^2 = i\langle Tx, x \rangle$. We also use (2.1) in the form $\|x\|_T^2 \leq \pi\|x\|^2$ and the parallelogram law:

$$\begin{aligned} \left| \sum_{n \neq m \leq N} \frac{a_n \overline{b_m}}{q_n - q_m} \right| &= \left| \frac{1}{4} \sum_{k=0}^3 i^k \|a + i^k b\|_T^2 \right| \\ &\leq \frac{1}{4} (\|a + b\|_T^2 + \|a - b\|_T^2 + \|a + ib\|_T^2 + \|a - ib\|_T^2) \\ &\leq \frac{\pi}{4} (\|a + b\|^2 + \|a - b\|^2 + \|a + ib\|^2 + \|a - ib\|^2) \\ &= \pi(\|a\|^2 + \|b\|^2) \end{aligned}$$

Replacing (a, b) by $(\lambda a, \lambda^{-1}b)$ and selecting $\lambda^2 = \|b\|/\|a\|$ yields the desired result. $\square$

Remark 2.2. By letting $N \rightarrow \infty$ in this proposition, the same inequalities hold for sequences of complex numbers $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, and a sequence of pairwise distinct integers $(q_n)_{n \in \mathbb{N}}$, provided that the series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge. This last condition ensures that the infinite sum on the left-hand side converges absolutely.

The first inequality can be generalized to real numbers q_n given a lower bound on $q_n - q_m$ for $n \neq m$ [40, Thm. 16], where the constant in the inequality depends on this lower bound. For a weighted version of (2.1), see [8, Corollary 2]. For both inequalities, a similar variant of (2.2) can be obtained with an analogous proof. The study of Hilbert-type inequalities is a mathematical subject in its own right. For a modern summary, we refer the interested reader to [3].

2.2. The mean value theorem for Dirichlet polynomials (integral form)

We are now ready to prove the main theorem of this chapter. We follow the argument in [22, Thm. 5.2], and the constant is made explicit by the author. The proof is divided into several steps. The first step sets the stage and quickly realizes that it suffices to prove the estimate

$$\sum_{n \neq m \leq N} \frac{a_n \overline{a_m}}{\log(n) - \log(m)} \ll \sum_{n \leq N} n |a_n|^2.$$

This looks very similar to the Hilbert inequality (2.1), and we follow with an argument similar to (2.3). This leaves us to bound the sum

$$\sum_{n \neq m \leq N} \int_0^1 a_n \overline{a_m} \frac{e^{2\pi i(\log(n) - \log(m))y}}{\log(n) - \log(m)} dy.$$

We consider a dyadic subdivision of the interval $[1, N]$ into $I_k = (2^{-k}N, 2^{1-k}N]$ and estimate each sum over $(n, m) \in I_k \times I_l$ individually for all $k, l \geq 1$. We consider the cases $|k - l| \geq 2$ and $|k - l| \leq 1$ separately in the second and third steps, respectively, where the latter case relies on Hilbert's inequality. We finish by collecting all estimates together in the final step.

Theorem 2.3. Let $a_1, \dots, a_N$ be complex numbers and $T \geq 0$, then

$$\int_0^T \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt = T \sum_{n \leq N} |a_n|^2 + O \left(\sum_{n \leq N} n |a_n|^2 \right). \quad (2.4)$$

In case we have a sequence of complex numbers $(a_n)_{n \in \mathbb{N}}$ such that the series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} n |a_n|^2$ converge, then

$$\int_0^T \left| \sum_{n=1}^{\infty} a_n n^{it} \right|^2 dt = T \sum_{n=1}^{\infty} |a_n|^2 + O \left(\sum_{n=1}^{\infty} n |a_n|^2 \right). \quad (2.5)$$

In both cases, the constant in the O -term is absolute. In fact, the constant 2440 is good enough.

Proof. **Step I: Set-up** We first work out the left-hand side in the finite case:

$$\begin{aligned} \int_0^T \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt &= \int_0^T \left(\sum_{n \leq N} a_n n^{it} \right) \overline{\left(\sum_{m \leq N} a_m m^{it} \right)} dt \\ &= \sum_{n \leq N} a_n \overline{a_n} \int_0^T 1 dt + \sum_{n \neq m \leq N} a_n \overline{a_m} \int_0^T (n/m)^{it} dt \\ &= T \sum_{n \leq N} |a_n|^2 + \sum_{n \neq m \leq N} a_n \overline{a_m} \frac{(n/m)^{iT} - 1}{i(\log(n) - \log(m))}. \end{aligned} \quad (2.6)$$

Hence, for the finite case, it suffices to prove the estimate

$$\sum_{n \neq m \leq N} \frac{a_n \overline{a_m}}{\log(n) - \log(m)} \ll \sum_{n \leq N} n |a_n|^2.$$

Indeed, once this estimate is established, it can be applied in (2.6) directly, and a second time with a_n replaced by $a_n n^{iT}$.

Once this is proven, we immediately obtain the infinite case as well: The estimate

$$\left| \int_0^T \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt - T \sum_{n \leq N} |a_n|^2 \right| \ll \sum_{n=1}^{\infty} n |a_n|^2,$$

follows, and now we simply let $N \rightarrow \infty$. The conditions ensure that all series converge, and for the integral we use the dominated convergence theorem, where we dominate the integrand using the trivial inequality

$$\left| \sum_{n \leq N} a_n n^{it} \right|^2 \leq \left(\sum_{n=1}^{\infty} |a_n| \right)^2.$$

The last term is a finite constant and therefore integrable over $[0, T]$.

For simplicity of notation, write

$$G = \sum_{n \neq m \leq N} \frac{a_n \overline{a_m}}{\log(n) - \log(m)}.$$

Similarly as in the beginning of the previous proof, $\overline{G} = -G$, so G is purely imaginary. In the remainder of this argument, we will assume that $G/i \geq 0$. The case where $G/i < 0$ is analogous, but with $|\sum_{n \leq N} a_n e^{2\pi i \log(n)x}|^2$ replaced by $|\sum_{n \leq N} a_n e^{-2\pi i \log(n)x}|^2$ in the starting point of the next derivation. We compute

$$\begin{aligned} 0 &\leq \int_0^1 \int_0^y \left| \sum_{n \leq N} a_n e^{2\pi i \log(n)x} \right|^2 dx dy \\ &= \int_0^1 \int_0^y \left(\sum_{n \leq N} a_n e^{2\pi i \log(n)x} \right) \left(\sum_{m \leq N} \overline{a_m} e^{2\pi i \log(m)x} \right) dx dy \\ &= \int_0^1 \int_0^y \sum_{n \neq m \leq N} a_n \overline{a_m} e^{2\pi i (\log(n) - \log(m))x} dx dy + \sum_{n \leq N} a_n \overline{a_n} \int_0^1 \int_0^y 1 dx dy \\ &= \int_0^1 \sum_{n \neq m \leq N} a_n \overline{a_m} \frac{e^{2\pi i (\log(n) - \log(m))y} - 1}{2\pi i (\log(n) - \log(m))} dy + \frac{1}{2} \sum_{n \leq N} |a_n|^2 \\ &= \int_0^1 \sum_{n \neq m \leq N} a_n \overline{a_m} \frac{e^{2\pi i (\log(n) - \log(m))y}}{2\pi i (\log(n) - \log(m))} dy - \frac{G}{2\pi i} + \frac{1}{2} \sum_{n \leq N} |a_n|^2. \end{aligned}$$

Plug in $G/i = |G|$ (in case $G/i < 0$, we have $G/i = -|G|$) to find

$$\begin{aligned} |G| &\leq \pi \sum_{n \leq N} |a_n|^2 + \left| \sum_{n \neq m \leq N} \int_0^1 a_n \overline{a_m} \frac{e^{2\pi i (\log(n) - \log(m))y}}{\log(n) - \log(m)} dy \right| \\ &\leq \pi \sum_{n \leq N} |a_n|^2 + \sum_{k, l \geq 1} \left| \sum_{\substack{(n, m) \in I_k \times I_l \\ n \neq m}} \int_0^1 a_n \overline{a_m} \frac{e^{2\pi i (\log(n) - \log(m))y}}{\log(n) - \log(m)} dy \right|, \end{aligned} \quad (2.7)$$

where we split the sum using the disjoint intervals $I_j = (2^{-j}N, 2^{1-j}N]$. The remainder of the proof consists of estimating each term in the final sum. We distinguish the cases $|k - l| \geq 2$ and $|k - l| \leq 1$.

Step II: The terms where $|k - l| \geq 2$ For this case, we will use the following preliminary inequality for $(m, n) \in I_k \times I_l$. Since $|k - l| \geq 2$, we have

$$|\log(n) - \log(m)| \geq \log(2^{-\min\{k, l\}} N) - \log(2^{1-\max\{k, l\}} N) = (|k - l| - 1) \log(2) \geq |k - l| \frac{\log(2)}{2}.$$

Combining this with the Cauchy-Schwarz inequality yields a bound for a single term in (2.7):

$$\begin{aligned} & \left| \sum_{\substack{(n, m) \in I_k \times I_l \\ n \neq m}} \int_0^1 \frac{a_n \overline{a_m} e^{2\pi i(\log(n) - \log(m))y}}{\log(n) - \log(m)} dy \right| \\ & \leq \sum_{(n, m) \in I_k \times I_l} \frac{|a_n a_m|}{|\log(n) - \log(m)|} \left| \frac{e^{2\pi i(\log(n) - \log(m))} - 1}{2\pi(\log(n) - \log(m))} \right| \\ & \leq \frac{1}{\pi} \sum_{(n, m) \in I_k \times I_l} \frac{|a_n a_m|}{(\log(n) - \log(m))^2} \\ & \leq \frac{4}{\pi \log^2(2) |k - l|^2} \sum_{(n, m) \in I_k \times I_l} |a_n a_m| \\ & \leq \frac{4}{\pi \log^2(2) |k - l|^2} \left(\sum_{(n, m) \in I_k \times I_l} |a_n|^2 \right)^{1/2} \left(\sum_{(n, m) \in I_k \times I_l} |a_m|^2 \right)^{1/2} \\ & = \frac{4}{\pi \log^2(2) |k - l|^2} (\#\mathbb{N} \cap I_l)^{1/2} (\#\mathbb{N} \cap I_k)^{1/2} \left(\sum_{n \in I_k} |a_n|^2 \right)^{1/2} \left(\sum_{m \in I_l} |a_m|^2 \right)^{1/2} \\ & \leq \frac{4}{\pi \log^2(2) |k - l|^2} (2^{1-l} N)^{1/2} (2^{1-k} N)^{1/2} \left(\sum_{n \in I_k} |a_n|^2 \right)^{1/2} \left(\sum_{m \in I_l} |a_m|^2 \right)^{1/2} \\ & = \frac{8}{\pi \log^2(2) |k - l|^2} \left(\sum_{n \in I_k} 2^{-k} N |a_n|^2 \right)^{1/2} \left(\sum_{m \in I_l} 2^{-l} N |a_m|^2 \right)^{1/2} \\ & \leq \frac{8}{\pi \log^2(2) |k - l|^2} \left(\sum_{n \in I_k} n |a_n|^2 \right)^{1/2} \left(\sum_{m \in I_l} m |a_m|^2 \right)^{1/2}. \end{aligned}$$

Summing over all (k, l) with $|k - l| \geq 2$ and applying the Cauchy-Schwarz inequality again gives the estimate we require:

$$\begin{aligned} & \sum_{|k-l| \geq 2} \left| \sum_{\substack{(n, m) \in I_k \times I_l \\ n \neq m}} \int_0^1 \frac{a_n \overline{a_m} e^{2\pi i(\log(n) - \log(m))y}}{\log(n) - \log(m)} dy \right| \\ & \leq \frac{8}{\pi \log^2(2)} \sum_{|k-l| \geq 2} \left(\frac{1}{|k - l|^2} \sum_{n \in I_k} n |a_n|^2 \right)^{1/2} \left(\frac{1}{|k - l|^2} \sum_{m \in I_l} m |a_m|^2 \right)^{1/2} \\ & \leq \frac{8}{\pi \log^2(2)} \left(\sum_{|k-l| \geq 2} \frac{1}{|k - l|^2} \sum_{n \in I_k} n |a_n|^2 \right)^{1/2} \left(\sum_{|k-l| \geq 2} \frac{1}{|k - l|^2} \sum_{m \in I_l} m |a_m|^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{8}{\pi \log^2(2)} \left(\sum_{k=1}^{\infty} \sum_{n \in I_k} n |a_n|^2 \sum_{|l-k| \geq 2} \frac{1}{|k-l|^2} \right)^{1/2} \left(\sum_{l=1}^{\infty} \sum_{m \in I_l} m |a_m|^2 \sum_{|k-l| \geq 2} \frac{1}{|k-l|^2} \right)^{1/2} \\
&= \frac{8}{\pi \log^2(2)} \left(\frac{\pi^2}{3} - 2 \right) \sum_{n \leq N} n |a_n|^2.
\end{aligned}$$

Step III: The terms where $|k-l| \leq 1$ For these terms, we will use another preliminary inequality. First note that by studying the derivative of $f(x) = \log(1+x) - x/3$, we see that this function is non-decreasing in $[0, 2]$ and non-increasing in $[2, 3]$. Since $f(0) = 0$ and $f(3) > 0$, we conclude that $\log(1+x) \geq x/3$ for $x \in [0, 3]$.

Suppose that $(n, m) \in I_k \times I_l$ with $n > m$. Note that $4m \geq 4 \cdot 2^{-l}N \geq 2^{1-k}N \geq n$, and therefore $(n-m)/m \leq 3$. Using the fact that $|k-l| \leq 1$, the previous two observations yield

$$\begin{aligned}
&\lfloor 2^{5-k}N \log(n) \rfloor - \lfloor 2^{5-k}N \log(m) \rfloor \\
&\geq 2^{5-k}N(\log(n) - \log(m)) - 1 = 2^{5-k}N \log\left(1 + \frac{n-m}{m}\right) - 1 \\
&\geq 2^{5-k}N \frac{n-m}{3m} - 1 \geq 2^{5-k}N \frac{n-m}{3 \cdot 2^{1-l}N} - 1 \geq 2^{5-k} \frac{n-m}{3 \cdot 2^{2-k}} - 1 \\
&= \frac{8}{3}(n-m) - 1 \\
&\geq n-m.
\end{aligned} \tag{2.8}$$

If $m > n$, then we similarly obtain the inequality $\lfloor 2^{5-k}N \log(m) \rfloor - \lfloor 2^{5-k}N \log(n) \rfloor \geq 2^{5-k}N(\log(m) - \log(n)) - 1 \geq m-n$.

We are now ready to bound the final terms in (2.7), for which it is enough to estimate the integrands:

$$\begin{aligned}
&\left| \sum_{\substack{(n,m) \in I_k \times I_l \\ n \neq m}} \frac{a_n e^{2\pi i \log(n)y} \overline{a_m e^{2\pi i \log(m)y}}}{\log(n) - \log(m)} \right| \\
&= \left| 2^{5-k}N \sum_{\substack{(n,m) \in I_k \times I_l \\ n \neq m}} \frac{a_n e^{2\pi i \log(n)y} \overline{a_m e^{2\pi i \log(m)y}}}{2^{5-k}N \log(n) - 2^{5-k}N \log(m)} \right| \\
&\leq \left| 2^{5-k}N \sum_{\substack{(n,m) \in I_k \times I_l \\ n \neq m}} \frac{a_n e^{2\pi i \log(n)y} \overline{a_m e^{2\pi i \log(m)y}}}{\lfloor 2^{5-k}N \log(n) \rfloor - \lfloor 2^{5-k}N \log(m) \rfloor} \right| \\
&\quad + 2^{5-k}N \sum_{\substack{(n,m) \in I_k \times I_l \\ n \neq m}} \frac{|a_n a_m|}{|\lfloor 2^{5-k}N \log(n) \rfloor - \lfloor 2^{5-k}N \log(m) \rfloor| |2^{5-k}N \log(n) - 2^{5-k}N \log(m)|}.
\end{aligned}$$

to bound the first term, we will use Hilbert's inequality (2.2) with $q_n = \lfloor 2^{5-k}N \log(n) \rfloor$,

which are pairwise different by (2.8). If $k \neq l$, we have to relabel the coefficients a_n . We define

$$a'_n = \begin{cases} a_n e^{2\pi i \log(n)y} & \text{if } n \in I_k, \\ 0 & \text{if } n \in I_l, \end{cases} \quad b'_m = \begin{cases} a_m e^{2\pi i \log(m)y} & \text{if } m \in I_l, \\ 0 & \text{if } m \in I_k, \end{cases}$$

and apply Hilbert's inequality, where we extend the sum over all $n, m \in I_k \cup I_l$, leading to the following estimate for the first term:

$$\begin{aligned} & 2\pi \cdot 2^{5-k} N \left(\sum_{n \in I_k} |a_n|^2 \right)^{1/2} \left(\sum_{m \in I_l} |a_m|^2 \right)^{1/2} \\ & \leq 2^{13/2} \pi \left(2^{-k} N \sum_{n \in I_k} |a_n|^2 \right)^{1/2} \left(2^{-l} N \sum_{m \in I_l} |a_m|^2 \right)^{1/2} \\ & \leq 2^{13/2} \pi \left(\sum_{n \in I_k} n |a_n|^2 \right)^{1/2} \left(\sum_{m \in I_l} m |a_m|^2 \right)^{1/2}. \end{aligned}$$

If $k = l$, then we can use the first inequality in Proposition 2.1 to obtain the same estimate, but where a factor 2 is removed.

For the second term, we use our preliminary bound (2.8) and the Cauchy-Schwarz inequality to estimate it by

$$\begin{aligned} & 2^{5-k} N \sum_{\substack{(n, m) \in I_k \times I_l \\ n \neq m}} \frac{|a_n a_m|}{(n - m)^2} \\ & \leq 2^{5-k} N \left(\sum_{\substack{(n, m) \in I_k \times I_l \\ n \neq m}} \frac{|a_n|^2}{(n - m)^2} \right)^{1/2} \left(\sum_{\substack{(n, m) \in I_k \times I_l \\ n \neq m}} \frac{|a_m|^2}{(n - m)^2} \right)^{1/2} \\ & = 2^{5-k} N \left(\sum_{n \in I_k} |a_n|^2 \sum_{\substack{m \in I_l \\ m \neq n}} \frac{1}{(n - m)^2} \right)^{1/2} \left(\sum_{m \in I_l} |a_m|^2 \sum_{\substack{n \in I_k \\ n \neq m}} \frac{1}{(n - m)^2} \right)^{1/2} \\ & \leq \frac{2^{11/2} \pi^2}{3} \left(2^{-k} N \sum_{n \in I_k} |a_n|^2 \right)^{1/2} \left(2^{-l} N \sum_{m \in I_l} |a_m|^2 \right)^{1/2} \\ & \leq \frac{2^{11/2} \pi^2}{3} \left(\sum_{n \in I_k} n |a_n|^2 \right)^{1/2} \left(\sum_{m \in I_l} m |a_m|^2 \right)^{1/2}. \end{aligned}$$

Summing over all (k, l) with $|k - l| \leq 1$, splitting the cases where $k = l$ and $k \neq l$, and using the Cauchy-Schwarz inequality in the latter case finally yields

$$\sum_{|k-l| \leq 1} \left| \sum_{\substack{(n, m) \in I_k \times I_l \\ n \neq m}} \int_0^1 a_n \overline{a_m} \frac{e^{2\pi i (\log(n) - \log(m))y}}{\log(n) - \log(m)} dy \right|$$

$$\begin{aligned}
&\leq \frac{2^{13/2}\pi^2}{3} \sum_{k=1}^{\infty} \sum_{n \in I_k} n|a_n|^2 + 2^{13/2}\pi^2 \sum_{k=1}^{\infty} \left(\sum_{n \in I_k} n|a_n|^2 \right)^{1/2} \left(\sum_{m \in I_{k+1}} m|a_m|^2 \right)^{1/2} \\
&\leq \frac{2^{13/2}\pi^2}{3} \sum_{n \leq N} n|a_n|^2 + 2^{13/2}\pi^2 \left(\sum_{k=1}^{\infty} \sum_{n \in I_k} n|a_n|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \sum_{m \in I_{k+1}} m|a_m|^2 \right)^{1/2} \\
&\leq \frac{2^{17/2}\pi^2}{3} \sum_{n \leq N} n|a_n|^2.
\end{aligned}$$

Step IV: Conclusion By combining the estimates for both cases of (k, l) , we conclude that

$$|G| \leq \left(\pi + \frac{8}{\pi \log^2(2)} \left(\frac{\pi^2}{3} - 2 \right) + \frac{2^{17/2}\pi^2}{3} \right) \sum_{n \leq N} n|a_n|^2.$$

In Step I of the proof, it is seen that the constant in the theorem is twice the one obtained here. By estimating $3 < \pi < 3.15$, $\sqrt{2} < 1.42$ and $1/\log^2(2) < 4$, a direct computation shows that it is smaller than 2440. $\square$

Remark 2.4. By replacing a_n with $a_n n^{iT_0}$ for some $T_0 \in \mathbb{R}$ in the above theorem, we see that the theorem still holds when we integrate over $[T_0, T_0 + T]$ instead of $[0, T]$.

2.3. The mean value theorem for Dirichlet polynomials (discrete form)

As a consequence of the integral variant of the mean value theorem for Dirichlet polynomials, we can prove its discrete variant. Instead of integrating, we now evaluate the integrand in a finite number of well-spaced values for t and estimate the sum. The idea of the proof is to use the well-spaced condition to bound the sum by an integral to which we can apply the previous theorem. We follow the proof in [22, Thm. 5.3].

Theorem 2.5. There exists an absolute constant $C > 0$ such that the following holds. Suppose $1 \leq t_1 < \dots < t_R \leq T$ are real numbers such that $|t_r - t_s| \geq 1$ for $r \neq s \leq R$. Let $a_1, \dots, a_N$ be arbitrary complex numbers. Then

$$\sum_{r \leq R} \left| \sum_{n \leq N} a_n n^{it_r} \right|^2 \leq C \left(T \sum_{n \leq N} |a_n|^2 + \sum_{n \leq N} n|a_n|^2 \right) \log(N+1). \quad (2.9)$$

Proof. Suppose $F \in C^1[0, 1]$ and define $f(x) = F(x + t_r - 1/2)$. Fix some $x \in [0, 1]$. Integration by parts shows

$$\begin{aligned}
f(x) &= \int_0^1 f(t) dt + \int_0^1 t f'(t) dt - (f(1) - f(x)) \\
&= \int_0^1 f(t) dt + \int_0^x t f'(t) dt + \int_x^1 (t-1) f'(t) dt.
\end{aligned}$$

In particular,

$$|f(1/2)| \leq \int_0^1 |f(t)| dt + \frac{1}{2} \int_0^1 |f'(t)| dt,$$

or equivalently

$$|F(t_r)| \leq \int_{t_r-1/2}^{t_r+1/2} |F(t)| dt + \frac{1}{2} \int_{t_r-1/2}^{t_r+1/2} |F'(t)| dt.$$

We apply this inequality to the function $F(t) = (\sum_{n \leq N} a_n n^{it})^2$. By the spacing condition imposed on the t_r , this yields

$$\begin{aligned} \sum_{r \leq R} \left| \sum_{n \leq N} a_n n^{it_r} \right|^2 &= \sum_{r \leq R} |F(t_r)| \\ &\leq \sum_{r \leq R} \int_{t_r-1/2}^{t_r+1/2} |F(t)| dt + \frac{1}{2} \sum_{r \leq R} \int_{t_r-1/2}^{t_r+1/2} |F'(t)| dt \\ &\leq \int_0^{T+1} |F(t)| dt + \frac{1}{2} \int_0^{T+1} |F'(t)| dt. \end{aligned}$$

To bound the first term, Theorem 2.3 can be applied directly to obtain

$$\int_0^{T+1} |F(t)| dt \ll T \sum_{n \leq N} |a_n|^2 + \sum_{n \leq N} n |a_n|^2.$$

The second term requires a bit more work. We directly calculate the derivative, apply the Cauchy–Schwarz inequality, and then use Theorem 2.3, yielding

$$\begin{aligned} &\frac{1}{2} \int_0^{T+1} |F'(t)| dt \\ &\leq \int_0^{T+1} \left| \sum_{n \leq N} a_n n^{it} \right| \left| \sum_{n \leq N} a_n n^{it} \log(n) \right| dt \\ &\leq \left(\int_0^{T+1} \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt \right)^{1/2} \left(\int_0^{T+1} \left| \sum_{n \leq N} a_n n^{it} \log(n) \right|^2 dt \right)^{1/2} \\ &\ll \left(T \sum_{n \leq N} |a_n|^2 + \sum_{n \leq N} n |a_n|^2 \right)^{1/2} \left(T \sum_{n \leq N} |a_n|^2 \log^2(n) + \sum_{n \leq N} n |a_n|^2 \log^2(n) \right)^{1/2} \\ &\leq \left(T \sum_{n \leq N} |a_n|^2 + \sum_{n \leq N} n |a_n|^2 \right) \log(N). \quad \square \end{aligned}$$

We conclude with two remarks on this theorem.

Remark 2.6. Similarly to Remark 2.4, the same theorem will hold if we consider any $T_0 \in \mathbb{R}$ and $T_0 + 1 \leq t_1 < \dots < t_R \leq T_0 + T$ instead. This can be seen by writing $a_n n^{it_r} = a_n n^{iT_0} \cdot n^{i(t_r - T_0)}$ and applying the theorem to $a_n n^{iT_0}$ instead of a_n and $t_r - T_0$ instead of t_r . As an alternative, we can also follow the same proof, but use Remark 2.4 when we apply Theorem 2.3.

Remark 2.7. As in Theorem 2.3, this result also has a variant with infinite sums. Namely, we have

$$\sum_{r \leq R} \left| \sum_{n=1}^{\infty} a_n n^{-it_r} \right|^2 \ll T \sum_{n=1}^{\infty} |a_n|^2 \log^2(n+1) + \sum_{n=1}^{\infty} n |a_n|^2 \log^2(n+1)$$

provided that

$$\sum_{n=1}^{\infty} |a_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n |a_n|^2 \log^2(n+1) < \infty.$$

The proof is the same, except that we ignore the last line and let $N \rightarrow \infty$ in the end.

Chapter 3

Exponential sums

One of the most important and difficult parts of analytic number theory is the estimation of so-called exponential sums. These sums take the form

$$\sum_{a < n \leq b} e(f(n)), \quad (3.1)$$

where f is a real-valued function satisfying certain smoothness conditions and $e(x) = e^{2\pi i x}$. The trivial estimate $\ll b - a$ of (3.1), which does not take into account any cancellation of the terms, is usually insufficient for applications. Therefore, more sophisticated methods are needed. In particular, it is interesting to consider estimates involving exponential integrals, as their cancellation is easier to deal with. This is the theme of the first section in this chapter, which serves to collect a few preliminary results.

In the context of exponential sums, special attention is given to (one-dimensional) exponent pairs (κ, λ) . Roughly speaking, these produce estimates of the form

$$\sum_{B < n \leq 2B} e(f(n)) \ll A^\kappa B^\lambda,$$

provided that

$$AB^{1-k} \ll |f^{(k)}(x)| \ll AB^{1-k} \quad (k = 1, \dots, m).$$

This idea essentially goes back to Van der Corput's method from the 1920s, which brought significant advancements in several classical problems of analytic number theory.

One of these applications, which is an essential ingredient in this thesis, concerns the theory of large values of the Riemann zeta function. This subject is closely related to bounding moments of the zeta function and therefore helps in obtaining zero-density estimates. We conclude this chapter with a brief discussion of this topic.

3.1. Preliminary lemmas

We begin by presenting a few preliminary results that directly estimate exponential sums, which will be useful later. In particular, we prove the following lemma, which was already used in Chapter 1 to estimate the zeta function in the critical strip (see Proposition 1.8).

It will also be applied to obtain our first crude zero-density estimate in Chapter 5. The proof follows [22, Lemma 1.2], where the constant is computed by the author.

Lemma 3.1. Suppose $f : [a, b] \rightarrow \mathbb{R}$ is a continuously differentiable function with monotonic derivative. Assume that $\sup_{a \leq x \leq b} |f'(x)| \leq \delta < 1$. Then

$$\left| \sum_{a < n < b} e(f(n)) - \int_a^b e(f(x)) dx \right| \leq \frac{10}{1 - \delta}. \quad (3.2)$$

Proof. We will assume that $\lfloor b \rfloor - \lfloor a \rfloor \geq 3$. In the other case, we trivially have (3.2) by the bound $|e(f(x))| \leq 1$. Let n be an integer in the interval $a < n \leq b - 1$. Define the function $\varphi_n(x) = e(f(n + x))$ for $0 < x < 1$ and

$$\varphi_n(0) = \frac{e(f(n)) + e(f(n + 1))}{2},$$

and extend it periodically to the real line with period 1. This allows us to write the function as a Fourier series of the form

$$\varphi_n(x) = \lim_{N \rightarrow \infty} \sum_{k=-N}^N \widehat{\varphi}_n(k) e(kx),$$

where

$$\widehat{\varphi}_n(0) = \int_0^1 e(f(n + x)) dx$$

and, if $k \neq 0$,

$$\begin{aligned} \widehat{\varphi}_n(k) &= \int_0^1 \varphi_n(x) e(-kx) dx \\ &= \left(\frac{e(f(n + x)) e(-kx)}{-2k\pi i} \right) \Big|_0^1 + \frac{1}{k} \int_0^1 f'(n + x) e(f(n + x) - kx) dx \\ &= \frac{e(f(n)) - e(f(n + 1))}{2k\pi i} + \frac{1}{k} \int_0^1 f'(n + x) e(f(n + x) - kx) dx. \end{aligned}$$

In particular,

$$\begin{aligned} &\frac{e(f(n)) + e(f(n + 1))}{2} \\ &= \varphi_n(0) \\ &= \lim_{N \rightarrow \infty} \sum_{k=-N}^N \widehat{\varphi}_n(k) \\ &= \int_0^1 e(f(n + x)) dx + \lim_{N \rightarrow \infty} \sum_{\substack{k=-N \\ k \neq 0}}^N \frac{1}{k} \int_0^1 f'(n + x) e(f(n + x) - kx) dx \end{aligned}$$

$$= \int_n^{n+1} e(f(x))dx + \lim_{N \rightarrow \infty} \sum_{\substack{k=-N \\ k \neq 0}}^N \frac{1}{k} \int_n^{n+1} f'(x)e(f(x) - kx)dx.$$

Summing this expression over the integers n with $a < n \leq b - 1$ yields

$$\sum_{a < n \leq b} e(f(n)) = \int_a^b e(f(x))dx + \lim_{N \rightarrow \infty} \sum_{\substack{k=-N \\ k \neq 0}}^N \frac{1}{k} \int_{[a]+1}^{[b]} f'(x)e(f(x) - kx)dx + O(1), \quad (3.3)$$

where the constant in the O -term is 3.

For $k \neq 0$, write

$$\int_{[a]+1}^{[b]} f'(x)e(f(x) - kx)dx = \frac{1}{2\pi i} \int_{[a]+1}^{[b]} \frac{f'(x)}{f'(x) - k} (e(f(x) - kx))' dx$$

and note that $f'(x)/(f'(x) - k)$ is a monotonic function because f' and $g : [-\delta, \delta] \rightarrow \mathbb{R} : x \mapsto x/(x - k)$ are. Therefore, we can apply the second mean value theorem for real integrals, to the real and imaginary part separately. For the real part, this asserts that there exists some $\xi \in ([a] + 1, [b])$ such that

$$\begin{aligned} & \left| \int_{[a]+1}^{[b]} \frac{f'(x)}{f'(x) - k} \Re(e(f(x) - kx))' dx \right| \\ &= \left| \frac{f'(a)}{f'(a) - k} \int_{[a]+1}^{\xi} \Re(e(f(x) - kx))' dx + \frac{f'(b)}{f'(b) - k} \int_{\xi}^{[b]} \Re(e(f(x) - kx))' dx \right| \\ &\leq \frac{4\delta}{|k| - \delta} \\ &\leq \frac{4}{|k| - \delta}, \end{aligned}$$

where the second to last inequality follows from the fact that $|g(x)|$ is maximal in $x = \pm\delta$. Similarly, we estimate the imaginary part by the same bound. In particular, we have

$$\begin{aligned} \left| \lim_{N \rightarrow \infty} \sum_{\substack{k=-N \\ k \neq 0}}^N \frac{1}{k} \int_{[a]+1}^{[b]} f'(x)e(f(x) - kx)dx \right| &\leq 2 \sum_{k=1}^{\infty} \frac{4/\pi}{k(k - \delta)} \\ &\leq \frac{8/\pi}{1 - \delta} + \frac{8}{\pi} \sum_{k=2}^{\infty} \frac{1}{(k - 1)^2} \\ &\leq \frac{8/\pi + 4\pi/3}{1 - \delta}. \end{aligned}$$

Estimating $3.14 \leq \pi \leq 3.15$ shows that the numerator is at most 7. Plugging the result into (3.3) proves the lemma. $\square$

We also mention the following two lemmas without proof. The interested reader may consult [22, Lemmas 2.2 and 2.4] for the arguments. We include them as they will be used in the next section to prove that $(1/2, 1/2)$ is an exponent pair.

Lemma 3.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable function. Suppose that there exists some $m > 0$ such that $f''(x) \geq m$ everywhere or $f''(x) \leq -m$ everywhere. Then

$$\left| \int_a^b e^{if(x)} dx \right| \leq 8m^{-1/2}.$$

Lemma 3.3. Suppose $f[a, b] \rightarrow \mathbb{R}$ is a twice continuously differentiable function such that $f''(x) < 0$ for all $x \in [a, b]$, and denote $\alpha = f'(b)$ and $\beta = f'(a)$. Then, for any η with $0 < \eta < 1$, we have

$$\sum_{a < n \leq b} e(f(n)) = \sum_{\alpha - \eta - 1 < m < \beta + \eta + 1} \int_a^b e(f(x) - mx) dx + O(\log(\beta - \alpha + 2)),$$

where the constant in the O -term is absolute.

3.2. Exponent pairs

In the following definition, we introduce the notion of exponent pairs. This is a concept that arises naturally in many problems in analytic number theory, in such a way that new exponent pairs immediately improve certain results. In particular, they play an important role in the study of large values of the Riemann zeta function, see e.g. Theorem 3.8 below. This subject is closely connected to moment estimates, which is why exponent pairs also appear in the theory of zero-density estimates. In fact, Section 5.5 below will obtain an estimate that still depends on the choice of an exponent pair, which allows us to optimize its choice over the set of newly discovered exponent pairs in [39]. Other applications include bounds for the order of the Riemann zeta functions on the critical strip (as in Theorem 1.10) and estimates for the error term in the Dirichlet divisor problem (see e.g. [39, Sections 2.4 and 5.4]).

To define exponent pairs, we follow [22, pp. 72-73]. The conditions imposed on f make the theory quite restrictive, but still allow us to study a large class of functions that occur in many important applications.

Definition 3.4 (Exponent pair). We call a pair of real numbers (κ, λ) with $0 \leq \kappa \leq 1/2 \leq \lambda \leq 1$ and $\kappa + \lambda \leq 1$ an exponent pair if the following condition holds:

There exists an integer $N \geq 1$ and real numbers $C > 1$, $D > 0$ such that if we are given

- $1 \leq h \leq B$,
- $A > 1/2$,
- an N -times continuously differentiable function $f : [B, B + h] \rightarrow \mathbb{R}$ with

$$AB^{1-n}/C \leq |f^{(n)}(x)| \leq CAB^{1-n} \quad (3.4)$$

for all $x \in [B, B + h]$ and $n = 1, \dots, N$,

then

$$\left| \sum_{B < n \leq B+h} e(f(n)) \right| \leq DA^\kappa B^\lambda. \quad (3.5)$$

The set of exponent pairs is convex. Indeed, suppose that (κ_1, λ_1) and (κ_2, λ_2) are exponent pairs and $0 \leq t \leq 1$. In Definition 3.4, take N to be the maximum out of the values used in both exponent pairs, and C the minimum. Denote the sum in (3.5) by S . Then

$$|S| = |S|^t |S|^{1-t} \ll A^{t\kappa_1 + (1-t)\kappa_2} B^{t\lambda_1 + (1-t)\lambda_2},$$

proving that $t(\kappa_1, \lambda_1) + (1-t)(\kappa_2, \lambda_2)$ is an exponent pair as well. Also remark that, trivially, $(0, 1)$ is an exponent pair. The machinery from the previous section allows us to easily obtain another exponent pair in the next proposition, which will directly be used to derive one of the zero-density estimates in Chapter 5.

Proposition 3.5. The pair $(1/2, 1/2)$ is an exponent pair.

Proof. Again, let S denote the sum in (3.5). Note that the sign of f'' is constant since it is continuous and non-zero by the lower bound $AB^{-1} \ll |f''(x)|$. By taking the complex conjugate of S , we may replace f with $-f$, so we can assume that $f''(x) < 0$ holds on $[B, B + h]$. In Definition 3.4, take $N = 2$ and any $C > 1$.

We apply Lemma 3.3 to S with $\eta = 1/2$:

$$S \ll \sum_{\alpha-3/2 < n < \beta+3/2} \int_a^b e(f(x) - nx) dx + \log(\beta - \alpha + 2).$$

Each integral can be estimated by Lemma 3.2 with $m \gg AB^{-1}$:

$$\begin{aligned} \sum_{\alpha-3/2 < n < \beta+3/2} \int_a^b e(f(x) - nx) dx &\ll (\beta - \alpha + 4) A^{-1/2} B^{1/2} \\ &\leq (|f'(B)| + |f'(B + h)| + 4) A^{-1/2} B^{1/2} \end{aligned}$$

$$\ll A^{1/2}B^{1/2},$$

where we used that $|f'(x)| \ll A$.

To estimate the logarithmic term, we require a preliminary inequality. Namely, we use the Taylor series of the exponential function to deduce, for $x > 0$,

$$x \leq 2 \frac{(x^{1/2})^2}{2} \leq 2e^{x^{1/2}},$$

so $\log(x) \leq \log(2) + x^{1/2}$. Note that by the conditions on f and since $A > 1/2$, there is some absolute constant $C > 0$ such that $\beta - \alpha + 2 \leq CA$. Therefore, using additionally that $B \geq 1$, we have

$$\begin{aligned} \log(\beta - \alpha + 2) &\leq \log(C) + \log(A) \\ &\leq (\sqrt{2}\log(C) + \sqrt{2}\log(2) + 1)A^{1/2}B^{1/2}. \end{aligned} \quad \square$$

In the recent paper [39], Trudgian and Yang summarized the entire set of known exponent pairs at that time. This is the set that we will consider in Chapter 5 when optimizing an exponent pair to improve current zero-density estimates. To describe this set, we consider the functions

$$\begin{aligned} A(x, y) &:= \left(\frac{x}{2x+2}, \frac{y}{2x+2} + \frac{1}{2} \right), \\ B(x, y) &:= \left(y - \frac{1}{2}, x + \frac{1}{2} \right). \end{aligned}$$

It is known (see e.g. [22, Lemmas 2.8 and 2.9]) that if (κ, λ) is an exponent pair, so are $A(\kappa, \lambda)$ and $B(\kappa, \lambda)$. The latter statement reflects the symmetry of the set of exponent pairs with respect to the line $y = x + 1/2$. In particular, these statements allow us to quickly obtain the exponent pair $(2/7, 4/7) = B(A(1/2, 1/2))$, which will be used multiple times in this thesis.

Generally, exponent pairs are considered better the further they lie towards the lower-left region of the triangle with vertices $(0, 1/2)$, $(0, 1)$, and $(1/2, 1/2)$. The best exponent pairs of [39] can be summarized in a sequence $((x_n, y_n))_{n \in \mathbb{Z}}$ as follows, where $\varepsilon > 0$ can attain any sufficiently small value. First, set

$$\begin{aligned} (x_0, y_0) &:= \left(\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon \right), & (x_1, y_1) &:= \left(\frac{4742}{38463} + \varepsilon, \frac{35731}{51284} + \varepsilon \right), \\ (x_2, y_2) &:= \left(\frac{18}{199} + \varepsilon, \frac{593}{796} + \varepsilon \right), & (x_3, y_3) &:= \left(\frac{2779}{38033} + \varepsilon, \frac{58699}{76066} + \varepsilon \right), \\ (x_4, y_4) &:= \left(\frac{715}{10238} + \varepsilon, \frac{7955}{10238} + \varepsilon \right). \end{aligned}$$

Bourgain [7, Thm. 6] proved the first exponent pair using the Bombieri–Iwaniec method [4, 5]. The others were proven by Trudgian and Yang [39, Lemma 1.1]. It follows from the previous paragraph that

$$(x_n, y_n) := A(x_{n-4}, y_{n-4})$$

is also an exponent pair for $n = 5, 6, 7, 8$. Next, for any $n \geq 9$, Heath-Brown [18] showed that

$$(x_n, y_n) := \left(\frac{2}{(n-5)^2(n-2)}, 1 - \frac{3n-14}{(n-4)(n-5)(n-2)} + \varepsilon \right)$$

is an exponent pair. Finally, due to symmetry, we have the exponent pairs

$$(x_n, y_n) := B(x_{-n}, y_{-n})$$

for $n \leq -1$.

The exponent pairs corresponding to $-20 \leq n \leq 20$ are plotted in Figure 3.1. Together with $(0, 1)$ and $(1/2, 1/2)$, their convex hull forms a set of known exponent pairs, symmetric with respect to the line $y = x + 1/2$. In this picture, we ignored the small value $\varepsilon > 0$, so we may only consider exponent pairs in the interior of this polygon, along with the line connecting $(0, 1)$ and $(1/2, 1/2)$. Recall that from Definition 3.4, we only consider pairs (κ, λ) within the triangle with vertices $(0, 1/2)$, $(0, 1)$, and $(1/2, 1/2)$. The exponent pair conjecture asserts that all elements of this triangle's interior are exponent pairs. Since $\zeta(1/2 + it) \ll_{\kappa, \lambda, \varepsilon} |t|^{\kappa + \lambda - 1/2 + \varepsilon}$ for any exponent pair (κ, λ) and $\varepsilon > 0$ (see e.g. [39, Section 5.4]), this would instantly imply the Lindelöf hypothesis.

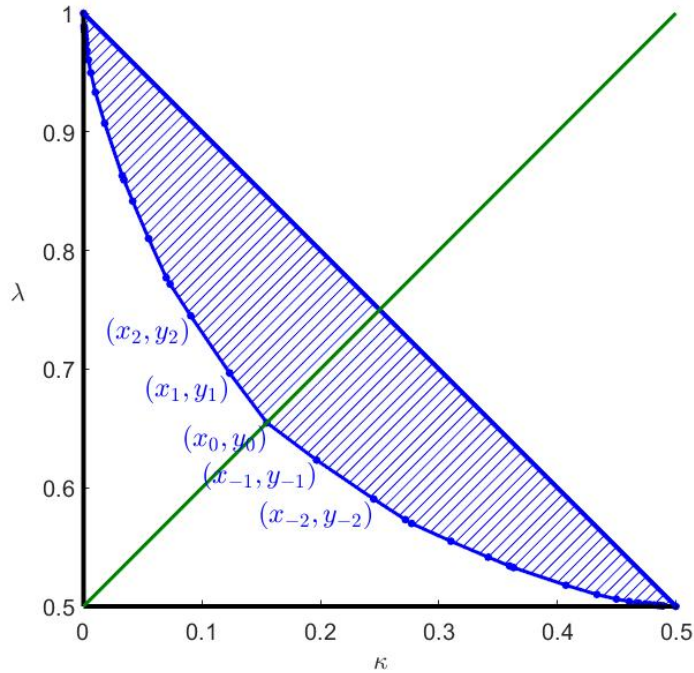


Figure 3.1: Known exponent pairs from [39] (in blue) and the line $y = x + 1/2$ (in green).

3.3. Application to large values of the Riemann zeta function

In this section, we consider two parameters $T \geq 2$ and $V > 0$ and real numbers $0 \leq t_1 \leq \dots \leq t_R \leq T$ ($R \geq 1$) with the well-spaced condition $|t_r - t_s| \geq 1$ for $r \neq s$. In

addition, we assume that they satisfy

$$|\zeta(1/2 + it_r)| \geq V.$$

This last condition motivates the name “large values”. The goal is to estimate R by an expression involving only T and V . Estimates of the form $R \ll f(T)V^{-b}$ are equivalent to moment estimates for the Riemann-zeta function on the critical line, see e.g. [22, pp. 201].

The following proposition provides a first estimate that will be used in one of our moment estimates of Chapter 4. It relies on a type of approximate functional equation of the zeta function, stated as a theorem below. Here, we introduce the k -fold divisor function

$$d_k(n) := \#\{(n_1, \dots, n_k) \in \mathbb{N}^k : n = n_1 \dots n_k\}. \quad (3.6)$$

Remark that $d_1(n) = 1$ is a constant function and $d_2(n)$ is the ordinary divisor function, often denoted by $d(n)$, which counts the number of positive divisors of n . The divisor functions will play a more important role in the next chapter, and appear in Chapter 5 as well. Also recall the notation $s = \sigma + it$ and the definition of $\chi(s)$ in (1.3).

Theorem 3.6 (Reflection Principle, [22, Thm. 4.4]). Prepare the following parameters:

- Fix $C > 0$ and an integer $k \geq 1$.
- Fix $0 < \alpha < 1$ and define $\delta_{\alpha, \sigma} = 1$ if $\alpha > \sigma$ and zero otherwise.
- Denote $h = \log^2(T)$ and suppose $1 \leq Y \leq T^C$ and $M \geq (3T)^k Y^{-1}$.

There exists some $T_0 \geq 2$ such that uniformly in $T \geq T_0$, $0 \leq \sigma \leq 1$ and $h^2 \leq t \leq T$, we have

$$\begin{aligned} \zeta^k(s) &= \sum_{n \leq 2Y} d_k(n) e^{-(n/Y)^h} n^{-s} + \delta_{\alpha, \sigma} \chi^k(s) \sum_{n \leq M} d_k(n) n^{s-1} \\ &\quad - \frac{1}{2\pi i} \int_{\alpha - \sigma - ih^2}^{\alpha - \sigma + ih^2} \chi^k(s+w) \left(\sum_{n \leq M} d_k(n) n^{w+s-1} \right) Y^w \Gamma\left(1 + \frac{w}{h}\right) \frac{dw}{w} + o(1), \end{aligned}$$

where the error term tends to zero if $T \rightarrow \infty$, independently of other parameters.

Proposition 3.7 ([22, Equation (8.26)]).

$$R \ll TV^{-4} \log^8(T) \quad (3.7)$$

Proof. We first reduce the problem to proving (3.7) when $T/2 \leq t_r \leq T$ for $r = 1, \dots, R$ and T is sufficiently large, say $T \geq T_0 \geq 2$. Once this is proven, apply it for $T, T/2, \dots, T/2^{k(T)}$, where $k(T)$ is the smallest non-negative integer such that $T/2^{k(T)} \geq T_0$. We sum the results to find

$$R \ll 2T_0 + 1 + \sum_{n=0}^{k(T)} \frac{T}{2^n} V^{-4} \log^8(T/2^n) \ll 2T_0 + 1 + TV^{-4} \log^8(T)$$

Note that, since $\mu(1/2) < 1/6$ by Theorem 1.10, we have $V \leq |\zeta(1/2 + it_r)| \ll t_r^{1/4} \leq T^{1/4}$, so $1 \ll TV^{-4} \log^8(T)$. In particular, we have (3.7).

Apply the previous theorem to $k = 2$, $Y = 3T$, $M = 3T$, $\alpha = 3/4$, and $s = 1/2 + it_r$. This yields

$$\begin{aligned} \zeta^2\left(\frac{1}{2} + it_r\right) &= \sum_{n \leq 6T} d(n) e^{-(n/3T)^h} n^{-1/2 - it_r} + \chi^2(1/2 + it_r) \sum_{n \leq 3T} d(n) n^{-1/2 + it_r} \\ &\quad - \frac{1}{2\pi i} \int_{\frac{1}{4} - ih^2}^{\frac{1}{4} + ih^2} \chi^2\left(\frac{1}{2} + it_r + w\right) \left(\sum_{n \leq 3T} d(n) n^{-\frac{1}{2} + it_r + w} \right) (3T)^w \Gamma\left(1 + \frac{w}{h}\right) \frac{dw}{w} \\ &\quad + o(1). \end{aligned}$$

Denote the j -th term on the right-hand side by $F_j(r)$ for $j = 1, 2, 3, 4$. Squaring both sides, summing over r , and using the Cauchy-Schwarz inequality yields

$$\begin{aligned} RV^4 &\leq \sum_{r \leq R} |\zeta(1/2 + it_r)|^4 \\ &\ll \sum_{r \leq R} \sum_{j=1}^4 |F_j(r)|^2 + \sum_{r \leq R} \sum_{k \neq j} |F_k(r)| |F_j(r)| \\ &\leq \sum_{j=1}^4 \sum_{r \leq R} |F_j(r)|^2 + \sum_{k \neq j} \left(\sum_{r \leq R} |F_k(r)|^2 \right)^{1/2} \left(\sum_{r \leq R} |F_j(r)|^2 \right)^{1/2}. \end{aligned}$$

Hence, it suffices to prove that $\sum_{r \leq R} |F_j(r)|^2 \ll T \log^8(T)$ for $j = 1, 2, 3, 4$.

In the case of $j = 4$, we trivially find

$$\sum_{r \leq R} |F_j(r)|^2 \ll R \ll T.$$

For $j = 1, 2$, we apply the mean value theorem for Dirichlet polynomials (see Theorem 2.5) as follows:

$$\begin{aligned} \sum_{r \leq R} |F_1(r)|^2 &\ll \left(T \sum_{n \leq 6T} \frac{d^2(n) e^{-2(n/3T)^h}}{n} + \sum_{n \leq 6T} d^2(n) e^{-2(n/3T)^h} \right) \log(T) \ll T \log^5(T), \\ \sum_{r \leq R} |F_2(r)|^2 &\ll \left(T \sum_{n \leq 3T} \frac{d^2(n)}{n} + \sum_{n \leq 3T} d^2(n) \right) \log(T) \ll T \log^5(T). \end{aligned}$$

Here, we used the elementary inequality $e^{-x} \leq 1$, the estimates of the sums $\sum_{n \leq x} d^2(n) n^a$ in Lemma 4.3 below, and the estimate of $\chi(s)$ in Corollary 1.3. Finally, if $j = 3$, we make the following deductions, whose individual steps are justified below:

$$\begin{aligned} &\sum_{r \leq R} |F_3(r)|^2 \\ &\ll \sum_{r \leq R} \left(\int_{-h^2}^{h^2} T^{-1/2} \left| \sum_{n \leq 3T} \frac{d(n)}{n^{1/4}} n^{it_r + iv} \right| T^{1/4} \left[1 + \left(\frac{|v|}{h} \right)^{1 + \frac{1}{2h}} \right]^{1/2} e^{-\frac{\pi}{2} \frac{|v|}{h}} \frac{1}{|\frac{1}{4} + iv|} dv \right)^2 \end{aligned}$$

$$\begin{aligned}
&\ll T^{-1/2} h^2 \sum_{r \leq R} \int_{-h^2}^{h^2} \left| \sum_{n \leq 3T} \frac{d(n)}{n^{1/4}} n^{it_r + iv} \right|^2 \frac{\left(1 + \left(\frac{|v|}{h}\right)^{1 + \frac{1}{2h}}\right) e^{-\pi|v|/h}}{\left|\frac{1}{4} + iv\right|^2} dv \\
&\ll T^{-1/2} \log^4(T) \left(T \sum_{n \leq 3T} \frac{d^2(n)}{n^{1/2}} + \sum_{n \leq 3T} d^2(n) n^{1/2} \right) \log(T) \\
&\quad \cdot \int_{-\infty}^{\infty} \frac{\left(1 + \frac{|v|}{h} + \left(\frac{|v|}{h}\right)^{3/2}\right) e^{-\pi|v|/h}}{v^2 + \frac{1}{16}} dv \\
&\ll T \log^8(T).
\end{aligned}$$

The successive steps are justified by the following results:

- Apply Stirling's formula in Theorem 1.2 and Corollary 1.3. For the Gamma function, we can extend the estimate to entire the strip $\{1 \leq \Re(s) \leq 5/4\}$ with the bound $|\Gamma(s)| \ll (|t|^{2\sigma-1} + 1)^{1/2} e^{-\pi|t|/2}$.
- Appeal to Jensen's inequality for convex functions (or the Cauchy-Schwarz inequality).
- Put the sum into the integral and exploit the mean value theorem for Dirichlet polynomials.
- Again use Lemma 4.3 below. In the integral, we estimate the numerator by a constant. $\square$

The following important theorem shows us how exponent pairs appear in this context. Namely, it is a key result in zero-density estimates since it implies a bound for the contribution of the class-II zeros. Moreover, we will exploit this fact to break current records for zero-density estimates. The proof relies on deeper results concerning the Riemann zeta function and is beyond the scope of this thesis. We therefore refer to [22, Thm. 8.2].

Theorem 3.8. Fix an exponent pair (κ, λ) . Then

$$R \ll TV^{-6} \log^8(T) + T^{\frac{\kappa+\lambda}{\kappa}} V^{-2 \frac{1+2\kappa+2\lambda}{\kappa}} (\log(T))^{\frac{3+6\kappa+4\lambda}{\kappa}}.$$

In particular, we apply this theorem to the exponent pairs $(2/7, 4/7)$ and $(1/2, 1/2)$ in a separate corollary that will be useful later. Part (i) will be used in a mixed moment estimate in our method that improves current zero-density estimates in Chapter 5. On the other hand, part (ii) will enable us to prove the current best bounds for moments of the zeta function on the critical line with exponents $4 \leq A \leq 12$.

Corollary 3.9.

(i)

$$R \ll TV^{-6} \log^8 T + T^3 V^{-19} \log^{49/2}(T),$$

(ii)

$$R \ll T^2 V^{-12} \log^{16}(T). \quad (3.8)$$

Proof. The first estimate follows immediately from Theorem 3.8 with the exponent pair $(2/7, 4/7)$. For the second estimate, use the exponent pair $(1/2, 1/2)$ to find

$$R \ll TV^{-6} \log^8(T) + (TV^{-6} \log^8(T))^2.$$

Again by Theorem 1.10, we have $V \leq |\zeta(1/2 + it_r)| \ll t_r^{1/6} \leq T^{1/6}$. In particular, since $R \neq 0$, we have $1 \ll TV^{-6} \log^8(T)$, so

$$R \ll (TV^{-6} \log^8(T))^2. \quad \square$$

Part II

**Theory of the Riemann Zeta
Function**

Chapter 4

Moment estimates on the critical line

After the preparatory work of the previous chapters, we begin to take a deeper dive into the theory of the Riemann zeta function in this chapter. In particular, we will cover power moments of the zeta function on the critical line $\Re(s) = 1/2$. These are expressions of the form

$$\int_0^T |\zeta(1/2 + it)|^A dt \tag{4.1}$$

as a function of $T \geq 0$, where $A \geq 0$ is a fixed real exponent. Theory of moments on other vertical lines exists as well, see e.g. [22, Chapter 8].

These moments naturally pop up in zero-density estimates when bounding a number of well-spaced class-II zeros. More specifically, a factor that appears here is a sum over well-spaced values of t evaluated in the integrand of (4.1). Estimating such sums is equivalent to estimating moments, as explained in Subsection 4.4.2 below. Obtaining good estimates for (4.1) therefore provides an easy bound for the contribution of the class-II zeros.

We begin by proving asymptotic formulas for the second and fourth moments with error terms. To date, these remain the only moments with a proven asymptotic formula. Afterwards, we briefly present the conjectured generalization to even powers. The chapter ends with a discussion of the more difficult real power moments. In this last section, we prove an almost optimal estimate for $0 \leq A \leq 4$ and the current best upper bound for $4 \leq A \leq 12$.

4.1. The second moment

As a warm-up, we begin with the proof of the second moment estimate. The result follows quite easily once we substitute the estimate of the zeta function in the critical strip (1.8) into the integral

$$\int_{T/2}^T |\zeta(1/2 + it)|^2 dt$$

and use the mean value theorem for Dirichlet polynomials. Then, we replace T by $T/2, T/2^2, \dots$ and add all the results.

Theorem 4.1 ([22, Thm. 1.11]). For $T \geq 2$,

$$\int_0^T |\zeta(1/2 + it)|^2 dt = T \log(T) + O(T).$$

Proof. In Proposition 1.8, we choose $\sigma_0 = \sigma = 1/2$, $x = T \geq 2$, and $T/2 \leq t \leq T$. This gives us

$$\zeta(1/2 + it) = \sum_{n \leq T} \frac{1}{n^{1/2+it}} + O\left(\frac{T^{1/2}}{t}\right) + O(T^{-1/2}) = \sum_{n \leq T} \frac{1}{n^{1/2+it}} + O(T^{-1/2}).$$

Therefore, by working out the square and using the triangle and Cauchy-Schwarz inequality, we get

$$\begin{aligned} \int_{T/2}^T |\zeta(1/2 + it)|^2 dt &= \int_{T/2}^T \left| \sum_{n \leq T} \frac{1}{n^{1/2+it}} \right|^2 dt + \int_{T/2}^T |O(T^{-1/2})|^2 dt \\ &\quad + O\left(\left(\int_{T/2}^T \left| \sum_{n \leq T} \frac{1}{n^{1/2+it}} \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{T/2}^T |T^{-1/2}|^2 dt \right)^{\frac{1}{2}} \right). \end{aligned}$$

The second term is trivially $\ll 1$.

For the first term, we apply the mean value theorem for Dirichlet polynomials. We also use the Euler summation formula to estimate the sum $\sum_{n \leq x} \frac{1}{n}$. This yields

$$\begin{aligned} \int_{T/2}^T \left| \sum_{n \leq T} \frac{1}{n^{1/2+it}} \right|^2 dt &= \int_0^T \left| \sum_{n \leq T} \frac{1}{n^{1/2}} n^{it} \right|^2 dt - \int_0^{T/2} \left| \sum_{n \leq T} \frac{1}{n^{1/2}} n^{it} \right|^2 dt \\ &= \frac{T}{2} \sum_{n \leq T} \frac{1}{n} + O\left(\sum_{n \leq T} 1 \right) \\ &= \frac{1}{2} T \log(T) + O(T). \end{aligned}$$

In summary, we have

$$\int_{T/2}^T |\zeta(1/2 + it)|^2 dt = \frac{1}{2} T \log(T) + O(T)$$

for $T \geq 2$.

Now, for $T \geq 2$, let $k(T)$ be the integer such that $\frac{T}{2^{k(T)}} < 2 \leq \frac{T}{2^{k(T)-1}}$. Then, the following computation finishes the proof:

$$\int_0^T |\zeta(1/2 + it)|^2 dt = \sum_{n=0}^{k(T)-1} \int_{T/2^{n+1}}^{T/2^n} |\zeta(1/2 + it)|^2 dt + \int_0^{T/2^{k(T)}} |\zeta(1/2 + it)|^2 dt$$

$$\begin{aligned}
&= \sum_{n=0}^{k(T)-1} \frac{1}{2} \frac{T}{2^n} \log \left(\frac{T}{2^n} \right) + O \left(\sum_{n=0}^{\infty} \frac{T}{2^n} \right) + O \left(\int_0^2 |\zeta(1/2 + it)|^2 dt \right) \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{T}{2^n} \log \left(\frac{T}{2^n} \right) - \frac{1}{2} \sum_{n=k(T)}^{\infty} \frac{T}{2^n} \log \left(\frac{T}{2^n} \right) + O(T) \\
&= \frac{T}{2} \sum_{n=0}^{\infty} \frac{\log(T)}{2^n} - \frac{T}{2} \sum_{n=0}^{\infty} \frac{\log(2^n)}{2^n} + O \left(\sum_{n=0}^{\infty} \frac{2}{2^n} \log \left(\frac{2}{2^n} \right) \right) + O(T) \\
&= T \log(T) + O(T). \quad \square
\end{aligned}$$

Early in the 20th century, Hardy and Littlewood [14] were the first to prove an asymptotic formula for the second moment. Their interest in estimating these moments was initially due to the connection to the Lindelöf hypothesis. A few years later, Littlewood improved this significantly to

$$\int_0^T |\zeta(1/2 + it)|^2 dt = T \log(T) + (2\gamma - 1 - \log(2\pi))T + E(T)$$

with γ denoting the Euler-Mascheroni constant and $E(T) \ll_{\varepsilon} T^{\frac{3}{4}+\varepsilon}$. Not long after this, the better estimate $E(T) \ll_{\varepsilon} T^{\frac{1}{2}+\varepsilon}$ was shown by Ingham [20]. In 1949, Atkinson proved an explicit formula for this error term $E(T)$, which is discussed in [22, Chapter 15].

4.2. The fourth moment

In the method we use to prove the fourth moment estimate, it will be crucial to have estimates for the sums

$$\sum_{n \leq x} d^2(n) n^a \quad \text{and} \quad \sum_{n > x} d^2(n) n^a,$$

where $a \geq -1$ in the first sum and $a < -1$ in the second sum. In particular, the special case $a = -1$ is responsible for the main term. To achieve this, it suffices to handle the sum where $a = 0$. Every other case then follows from an easy partial summation argument. We will first determine the Dirichlet series of $d^2(n)$ using the method described in [22, Section 1.7]. This allows us to write this arithmetic function as a Dirichlet convolution, to which we can apply the convolution method.

Lemma 4.2. If $\sigma > 1$, we have

$$\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

Proof. If $\Re(s) = \sigma > 1$, we have

$$\sum_{p^k} \frac{d^2(p^k)}{p^{k\sigma}} = \sum_{p^k} \frac{(k+1)^2}{p^{k\sigma}} \leq \sum_{p^k} \frac{4 \log^2(p^k)}{\log^2(2) p^{k\sigma}} \leq \frac{4}{\log^2(2)} \sum_{n=1}^{\infty} \frac{\log^2(n)}{n^{\sigma}} < \infty,$$

and since $d^2(n)$ is multiplicative, this Dirichlet series converges absolutely in s .

To prove the identity, we invoke the Euler product for $\sigma > 1$:

$$\frac{\zeta^4(s)}{\zeta(2s)} = \prod_p \frac{1 - p^{-2s}}{(1 - p^{-s})^4} = \prod_p \frac{1 + p^{-s}}{(1 - p^{-s})^3}.$$

The next step is to replace the denominator by its Taylor series. Since for $|z| < 1$, the series $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ converges absolutely, the product series converges too:

$$\frac{1}{(1-z)^3} = \sum_{n=0}^{\infty} a_n z^n,$$

where

$$a_m = \sum_{k_1+k_2+k_3=m} 1 = \sum_{k_1=0}^m \sum_{k_2=0}^{m-k_1} 1 = \frac{(m+2)(m+1)}{2}.$$

Insert $z = p^{-s}$ into this Taylor series to get

$$\begin{aligned} \frac{\zeta^4(s)}{\zeta(2s)} &= \prod_p (1 + p^{-s}) \sum_{m=0}^{\infty} \frac{(m+2)(m+1)}{2} p^{-ms} \\ &= \prod_p \left(\sum_{m=0}^{\infty} \frac{(m+2)(m+1)}{2} p^{-ms} + \sum_{m=0}^{\infty} \frac{(m+1)m}{2} p^{-ms} \right) \\ &= \prod_p \sum_{m=0}^{\infty} (m+1)^2 p^{-ms} \\ &= \prod_p \sum_{m=0}^{\infty} d^2(p^m) p^{-ms} \\ &= \sum_{n=0}^{\infty} \frac{d^2(n)}{n^s}. \end{aligned}$$

In the last equality, we used the fact that $d(n)$, and therefore its square, is multiplicative, so we may write its Dirichlet series as an Euler product as well. $\square$

Lemma 4.3. For $x \geq 2$,

$$\sum_{n \leq x} d^2(n) n^a = \begin{cases} \frac{1}{\pi^2(a+1)} x^{1+a} \log^3(x) + O_a(x^{1+a} \log^2(x)) & \text{if } a > -1, \\ \frac{1}{4\pi^2} \log^4(x) + O(\log^3(x)) & \text{if } a = -1. \end{cases} \quad (4.2)$$

In case $a < -1$, we have for $x \geq 2$,

$$\sum_{n > x} d^2(n) n^a = \frac{-1}{\pi^2(a+1)} x^{1+a} \log^3(x) + O_a(x^{1+a} \log^2(x)). \quad (4.3)$$

Proof. First, define the arithmetic functions

$$d_4(n) = (d * d)(n), \quad f(n) = \begin{cases} \mu(\sqrt{n}) & \text{if } n \text{ is a square,} \\ 0 & \text{else.} \end{cases}$$

with the Dirichlet convolution. In the case of $d_4(n)$, this agrees with the definition of $d_k(n)$ in (3.6). The previous lemma tells us that $\zeta^4(s)/\zeta(2s)$, which is the Dirichlet series of $d_4 * f$, is also the Dirichlet series of d^2 . Therefore, since an arithmetic function uniquely defines its Dirichlet series, $d^2 = d_4 * f$.

Next, we apply the convolution method to $d = 1 * 1$, $d_4 = d * d$, and $d^2 = d_4 * f$. In between, we also use partial summation:

$$\sum_{n \leq x} d(n) = \sum_{k \leq x} \sum_{n \leq x/k} 1 = \sum_{k \leq x} \left(\frac{x}{k} + O(1) \right) = x \log(x) + O(x),$$

$$\begin{aligned} \sum_{n \leq x} d_4(n) &= \sum_{k \leq x} d(k) \left(\frac{x}{k} \log \left(\frac{x}{k} \right) + O \left(\frac{x}{k} \right) \right) \\ &= x \log(x) \left(\frac{1}{2} \log^2(x) + O(\log(x)) \right) - x \left(\frac{1}{3} \log^3(x) + O(\log^2(x)) \right) + O(x \log^2(x)) \\ &= \frac{1}{6} x \log^3(x) + O(x \log^2(x)), \end{aligned}$$

$$\begin{aligned} \sum_{n \leq x} d^2(n) &= \sum_{k \leq x} f(k) \left(\frac{1}{6} \frac{x}{k} \log^3 \left(\frac{x}{k} \right) + O \left(\frac{x}{k} \log^2 \left(\frac{x}{k} \right) \right) \right) \\ &= \frac{1}{6} x \log^3(x) \sum_{k \leq \sqrt{x}} \frac{\mu(k)}{k^2} - \frac{1}{2} x \log^2(x) \sum_{k \leq \sqrt{x}} \frac{\mu(k) \log(k^2)}{k^2} \\ &\quad + \frac{1}{2} x \log(x) \sum_{k \leq \sqrt{x}} \frac{\mu(k) \log^2(k^2)}{k^2} - \frac{1}{6} x \sum_{k \leq \sqrt{x}} \frac{\mu(k) \log^3(k^2)}{k^2} \\ &\quad + O \left(x \log^2(x) \sum_{k \leq x} \frac{|f(k)|}{k} \right) \\ &= \frac{1}{6} x \log^3(x) \left(\frac{1}{\zeta(2)} + O \left(\sum_{k > \sqrt{x}} \frac{1}{k^2} \right) \right) + O(x \log^2(x)) + O \left(x \log^2(x) \sum_{k \leq \sqrt{x}} \frac{1}{k^2} \right) \\ &= \frac{1}{\pi^2} x \log^3(x) + O(x \log^2(x)). \end{aligned} \tag{4.4}$$

We finish with an application of the summation by parts formula. □

The estimates appearing in this proof can be significantly improved with the Dirichlet hyperbola method. When applied to $\sum_{n \leq x} d(n)$, this argument gives an estimate with error term $O(x^{\frac{1}{2}})$, as explained in [38, Section 12.1]. For $\sum_{n \leq x} d_4(n)$, this method reduces us to the error term $O(x^{\frac{3}{4}} \log(x))$. The same convolution argument for $\sum_{n \leq x} d^2(n)$, but with a more careful treatment of all terms, gives the same error term. Then, we can easily, but with a more technical partial summation argument, improve the estimates in (4.2) and (4.3). However, the current statement will be enough for us to prove the following fourth moment estimate.

The proof will happen in four steps. The first step reduces the problem to estimating the integral $\int_{T/2}^T |\zeta(1/2 + it)|^4 dt$, similarly as when proving the second moment estimate. The next step derives an equation that helps us write $|\zeta(1/2 + it)|^2$ as a sum of six terms. Here, the starting point will be the identity

$$e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s} ds, \quad (4.5)$$

which holds for all $c, x > 0$. This equation follows from applying the Mellin inversion formula (see e.g. [2, Section 8.2]) to $\Gamma(s)$, which is the Mellin transform of the function e^{-x} by definition. The expression is surprisingly useful, as it will also serve as the entry point in the zero-detection method explained in Chapter 5. Moreover, the proof of the reflection principle (see Theorem 3.6) begins with this formula as well. In the third step, we square this equation, expand it, plug the result into the integral, and estimate the main term. The final step bounds all other terms.

Theorem 4.4. For $T \geq 2$,

$$\int_0^T |\zeta(1/2 + it)|^4 dt = \frac{1}{2\pi^2} T \log^4(T) + O(T \log^3(T)). \quad (4.6)$$

Proof. **Step I: Preliminary reduction** It suffices to prove that, for T sufficiently large,

$$\int_{T/2}^T |\zeta(1/2 + it)|^4 dt = \frac{1}{4\pi^2} T \log^4(T) + O(T \log^3(T)).$$

Indeed, suppose that this holds for $T \geq T_0$, where $T_0 \geq 2$ is a fixed real number. Take $k(T) \in \mathbb{N}$ such that $\frac{T}{2^{k(T)}} < T_0 \leq \frac{T}{2^{k(T)-1}}$. Then

$$\begin{aligned} & \int_0^T |\zeta(1/2 + it)|^4 dt \\ &= \sum_{n=0}^{k(T)-1} \int_{T/2^{n+1}}^{T/2^n} |\zeta(1/2 + it)|^4 dt + \int_0^{T/2^{k(T)}} |\zeta(1/2 + it)|^4 dt \\ &= \sum_{n=0}^{k(T)-1} \frac{1}{4\pi^2} \frac{T}{2^n} \log^4\left(\frac{T}{2^n}\right) + O\left(\sum_{n=0}^{\infty} \frac{T}{2^n} \log^3(T)\right) + O\left(\int_0^{T_0} |\zeta(1/2 + it)|^4 dt\right) \\ &= \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \left(\frac{T}{2^n} \log^4\left(\frac{T}{2^n}\right)\right) - \frac{1}{4\pi^2} \sum_{n=k(T)}^{\infty} \left(\frac{T}{2^n} \log^4\left(\frac{T}{2^n}\right)\right) + O(T \log^3(T)) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \left(\frac{T}{2^n} \log^4(T) \right) + O \left(\sum_{n=0}^{\infty} \left(\frac{T_0}{2^n} \log^4 \left(\frac{T_0}{2^n} \right) \right) \right) + O(T \log^3(T)) \\
&= \frac{1}{2\pi^2} T \log^4(T) + O(T \log^3(T)),
\end{aligned}$$

where, to obtain the fourth equality, we expanded the product $\log^4(T/2^n) = (\log(T) - \log(2^n))^4$ in the first sum and absorbed everything, except for the leading term, into the error term $O(T \log^3(T))$. Lastly, by continuity in T , this result holds for all $T \geq 2$.

Step II: Set-up We begin with a computation, whose individual steps are justified below. For convenience, we will assume that T is sufficiently large throughout the remainder of the proof. Let $s = 1/2 + it$ and $T/2 \leq t \leq T$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d(n)}{n^s} e^{-n/T} &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta^2(s+w) \Gamma(w) T^w dw \\
&= \zeta^2(s) + O_c(T^{-c}) + \frac{1}{2\pi i} \int_{-3/4-i\infty}^{-3/4+i\infty} \zeta^2(s+w) \Gamma(w) T^w dw \\
&= \zeta^2(s) + O_c(T^{-c}) + \frac{1}{2\pi i} \int_{-3/4-i\infty}^{-3/4+i\infty} \chi^2(s+w) \sum_{n=1}^{\infty} \frac{d(n)}{n^{1-s-w}} \Gamma(w) T^w dw \\
&= \zeta^2(s) + O_c(T^{-c}) + \frac{1}{2\pi i} \int_{-3/4-i\infty}^{-3/4+i\infty} \chi^2(s+w) \sum_{n>T} \frac{d(n)}{n^{1-s-w}} \Gamma(w) T^w dw \\
&\quad - \chi^2(s) \sum_{n \leq T} \frac{d(n)}{n^{1-s}} + \frac{1}{2\pi i} \int_{1/4-i\infty}^{1/4+i\infty} \chi^2(s+w) \sum_{n \leq T} \frac{d(n)}{n^{1-s-w}} \Gamma(w) T^w dw \\
&= \zeta^2(s) + O_c(T^{-c}) + \frac{1}{2\pi i} \int_{-3/4-i\log^2(T)}^{-3/4+i\log^2(T)} \chi^2(s+w) \sum_{n>T} \frac{d(n)}{n^{1-s-w}} \Gamma(w) T^w dw \\
&\quad - \chi^2(s) \sum_{n \leq T} \frac{d(n)}{n^{1-s}} + \frac{1}{2\pi i} \int_{1/4-i\log^2(T)}^{1/4+i\log^2(T)} \chi^2(s+w) \sum_{n \leq T} \frac{d(n)}{n^{1-s-w}} \Gamma(w) T^w dw.
\end{aligned}$$

We now explain each equality in order:

- Substitute (4.5) with $c = 2$, apply the dominated convergence theorem, and use the fact that $\zeta^2(s)$ is the Dirichlet series of $d(n)$. With the estimates of the Gamma function and the Riemann zeta function in Theorems 1.2 and 1.9, we obtain an integrable dominating function.
- Write the integral as a limit $\lim_{a \rightarrow \infty} \int_{2-ia}^{2+ia} \zeta^2(s+w) \Gamma(w) T^w dw$ and apply the residue theorem to the closed integration path that connects the points $2 + ia$, $-3/4 + ia$, $-3/4 - ia$ and $2 - ia$ with linear segments. Utilize the estimates for the Gamma function and the Riemann zeta function again to show that the integrals over the horizontal parts tend to 0 if $a \rightarrow \infty$.

The only poles to consider are $w = 0$ and $w = 1 - s$, with respective residues

$\zeta^2(s)$ and

$$\Gamma(1-s)T^{1-s} \left(2\gamma + \log(T) + \frac{\Gamma'(1-s)}{\Gamma(1-s)} \right).$$

We can estimate the latter residue by $O_c(T^{-c})$ for every $c > 0$. To see this, use Stirling's formula in Theorem 1.2 with $T/2 \leq t \leq T$ to bound the first factor. To bound the digamma function, apply the identity

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log(z) - \frac{1}{2z} - 2 \int_0^\infty \frac{x dx}{(x^2 + z^2)(e^{2\pi x} - 1)}$$

from [42, Example 12.3.7]. The only difficulty here lies in estimating the integral for $z = 1 - s = 1/2 - it$. However, when studying $|x^2 + z^2|^2$ as a function of t , after some easy calculations we see that it is minimal in $t^2 = x^2 - 1/4$ if $x \geq 1/2$, and in $t = 0$ otherwise. This yields $|x^2 + z^2| \geq x$ and $|x^2 + z^2| \geq x^2 + 1/4$ respectively. Therefore, we may bound the integral by a constant independent of t . Conclude that

$$\frac{\Gamma'(1-s)}{\Gamma(1-s)} \ll \log(t) \leq \log(T).$$

- Substitute the functional equation (1.4) and write $\zeta^2(1-s-w)$ as its Dirichlet series.
- Split the sum in the integral, and for the integral over the finite sum, change the path of integration from $\Re(w) = -3/4$ to $\Re(w) = 1/4$. This is done similarly as to obtain the second equality, but the path of integration is now clockwise. The only pole is in $w = 0$ and, unlike before, we also use the estimate for $\chi(s)$ in Corollary 1.3 to bound the integrals over the horizontal parts.
- Restrict the integral to the part where $|\Im(w)| \leq \log^2(T)$ and use Corollary 1.3 and Theorem 1.2 to estimate the tails by $O_c(T^{-c})$, for any $c > 0$.

We now multiply this computation with $\chi^{-1}(\frac{1}{2} + it)$, noting that, using the functional equation (1.4),

$$\frac{\zeta^2(\frac{1}{2} + it)}{\chi(\frac{1}{2} + it)} = \zeta(1/2 + it) \frac{\zeta(1 - (\frac{1}{2} - it))}{\chi(1 - (\frac{1}{2} - it))} = \overline{\zeta(1/2 + it)} \zeta(1/2 - it) = |\zeta(1/2 - it)|^2.$$

By the estimate of $\chi(s)$ in Corollary 1.3, this multiplication does not affect the error term $O_c(T^{-c})$. The resulting equation is

$$|\zeta(1/2 + it)|^2 = \sum_{k=1}^6 J_k(1/2 + it) + O_c(T^{-c}) \quad (4.7)$$

for all $c > 0$, where we introduced the functions

$$J_1(s) = \chi^{-1}(s) \sum_{n \leq T} \frac{d(n)}{n^s},$$

$$J_2(s) = \chi(s) \sum_{n \leq T} \frac{d(n)}{n^{1-s}},$$

$$\begin{aligned}
J_3(s) &= \chi^{-1}(s) \sum_{n>T} \frac{d(n)e^{-n/T}}{n^s}, \\
J_4(s) &= \chi^{-1}(s) \sum_{n \leq T} \frac{d(n)(e^{-n/T} - 1)}{n^s}, \\
J_5(s) &= -\frac{1}{2\pi i} \chi^{-1}(s) \int_{-3/4-i\log^2(T)}^{-3/4+i\log^2(T)} \chi^2(s+w) \sum_{n>T} \frac{d(n)}{n^{1-s-w}} \Gamma(w) T^w dw, \\
J_6(s) &= -\frac{1}{2\pi i} \chi^{-1}(s) \int_{1/4-i\log^2(T)}^{1/4+i\log^2(T)} \chi^2(s+w) \sum_{n \leq T} \frac{d(n)}{n^{1-s-w}} \Gamma(w) T^w dw,
\end{aligned}$$

which are all analytic in $0 < \Re(s) < 3/4$ (for $J_5(s)$ and $J_6(s)$, see e.g. [33, Chapter 2, Thm. 5.4]). Note here that $J_1(s)$ cancels with part of $J_4(s)$ when summed together.

Step III: The main term

We first show that $J_k(1/2 + it) \ll T^{1/2} \log(T)$, so when multiplied with the error term $O_c(T^{-c})$, we get an error term $O(1)$ for each k if we choose $c > 0$ big enough.

Use the estimate of $\chi(s)$ in Corollary 1.3, the bounds $e^{-x} \ll 1/x^2$ and $e^{-n/T} \leq 1$, and partial summation to show that $J_k(1/2 + it) \ll T^{1/2} \log(T)$ for $k = 1, 2, 3, 4$. To estimate $J_5(1/2 + it)$ and $J_6(1/2 + it)$, use Corollary 1.3 and partial summation together with Stirling's formula in Theorem 1.2:

$$\begin{aligned}
J_5(1/2 + it) &\ll \left(\int_{-\log^2(T)}^{-1} + \int_1^{\log^2(T)} \right) \left(|t + v|^{2 \cdot \frac{3}{4}} T^{-\frac{1}{4}} \log(T) |v|^{-\frac{5}{4}} e^{-\frac{\pi}{2}|v|} T^{-\frac{3}{4}} \right) dv \\
&\quad + \int_{-1}^1 |t + v|^{2 \cdot \frac{3}{4}} T^{-\frac{1}{4}} \log(T) |\Gamma(-3/4 + iv)| T^{-\frac{3}{4}} dv \\
&\ll T^{\frac{1}{2}} \log(T).
\end{aligned}$$

Bounding $J_6(1/2 + it)$ is analogous.

Before we insert (4.7) into the moment integral, use the functional equation (1.4) to see that

$$\overline{\chi(1/2 + it)} = \chi(1/2 - it) = \frac{\zeta(1/2 - it)}{\zeta(1/2 + it)} = \chi^{-1}(1/2 + it), \quad (4.8)$$

and therefore $J_1(1/2 + it) = \overline{J_2(1/2 + it)}$. Now, fixing a sufficiently large c , squaring and integrating (4.7) gives

$$\begin{aligned}
&\int_{T/2}^T |\zeta(1/2 + it)|^4 dt \\
&= 2 \int_{T/2}^T |J_1(1/2 + it)|^2 dt + O \left(\left| \int_{T/2}^T (J_1^2(1/2 + it) + J_2^2(1/2 + it)) dt \right| \right) \\
&\quad + O \left(\sum_{k=3}^6 \int_{T/2}^T |J_k(1/2 + it)|^2 dt \right) + O \left(\sum_{3 \leq k < l \leq 6} \int_{T/2}^T |J_k(1/2 + it)| |J_l(1/2 + it)| dt \right)
\end{aligned}$$

$$+ O\left(\sum_{k=3}^6 \left| \int_{T/2}^T (J_1(1/2 + it) + J_2(1/2 + it)) J_k(1/2 + it) dt \right| \right) + O(1). \quad (4.9)$$

For the first term, we use the fact that $|\chi(1/2 + it)| = 1$ from (4.8), the mean value theorem for Dirichlet polynomials, and the estimates for $\sum_{n \leq x} d^2(n) n^{-a}$ in Lemma 4.3:

$$\begin{aligned} 2 \int_{T/2}^T |J_1(1/2 + it)|^2 dt &= 2 \int_0^T \left| \sum_{n \leq T} \frac{d(n)}{n^{1/2}} n^{it} \right|^2 dt - 2 \int_0^{T/2} \left| \sum_{n \leq T} \frac{d(n)}{n^{1/2}} n^{it} \right|^2 dt \\ &= T \sum_{n \leq T} \frac{d^2(n)}{n} + O\left(\sum_{n \leq T} d^2(n)\right) \\ &= \frac{1}{4\pi^2} T \log^4(T) + O(T \log^3(T)). \end{aligned}$$

Step IV: Estimating the remaining terms Using the same methods as for the first term, while also using the estimate for $\chi(s)$ in Corollary 1.3, $e^{-x} \ll 1/x^2$ for $x \geq 1$ and $1 - e^{-x} \leq x$ for all $x \in \mathbb{R}$, we estimate in the range $0 < \sigma < 3/4$:

$$\begin{aligned} \int_{T/2}^T |J_3(\sigma + it)|^2 dt &\ll T^{2\sigma-1} \left(T \sum_{n>T} \frac{d^2(n)}{n^{2\sigma}} e^{-2n/T} + \sum_{n>T} \frac{d^2(n)}{n^{2\sigma-1}} e^{-2n/T} \right) \\ &\ll T^{2\sigma+2} \sum_{n>T} \frac{d^2(n)}{n^{2\sigma+2}} + T^{2\sigma+1} \sum_{n>T} \frac{d^2(n)}{n^{2\sigma+1}} \ll_{\sigma} T \log^3(T) \end{aligned}$$

and

$$\begin{aligned} \int_{T/2}^T |J_4(\sigma + it)|^2 dt &\ll T^{2\sigma-1} \left(T \sum_{n \leq T} \frac{d^2(n)}{n^{2\sigma}} (e^{-n/T} - 1)^2 + \sum_{n \leq T} \frac{d^2(n)}{n^{2\sigma-1}} (e^{-n/T} - 1)^2 \right) \\ &\leq T^{2\sigma-2} \sum_{n \leq T} \frac{d^2(n)}{n^{2\sigma-2}} + T^{2\sigma-3} \sum_{n \leq T} \frac{d^2(n)}{n^{2\sigma-3}} \ll T \log^3(T). \end{aligned}$$

In the case of $J_5(s)$ and $J_6(s)$ however, we need to employ the generalized Minkowski integral inequality (see e.g. [15, Thm. 202]) before we can appeal to Theorem 2.3. Before this, we first use Corollary 1.3, again for $0 < \sigma < 3/4$. This yields

$$\begin{aligned} \int_{T/2}^T |J_5(\sigma + it)|^2 dt &\ll \int_{T/2}^T T^{2\sigma-1} \left(\int_{-\log^2(T)}^{\log^2(T)} T^{\frac{5}{2}-2\sigma} \left| \sum_{n>T} \frac{d(n)}{n^{7/4-\sigma-it-iv}} \right| |\Gamma(-3/4 + iv)| T^{-\frac{3}{4}} dv \right)^2 dt \\ &\leq T^{\frac{5}{2}-2\sigma} \left(\int_{-\log^2(T)}^{\log^2(T)} \left(\int_{T/2}^T \left| \sum_{n>T} \frac{d(n)}{n^{7/4-\sigma-it-iv}} \right|^2 |\Gamma(-3/4 + iv)|^2 dt \right)^{\frac{1}{2}} dv \right)^2 \end{aligned}$$

$$\begin{aligned}
&\ll T^{\frac{5}{2}-2\sigma} \left(\int_{-\log^2(T)}^{\log^2(T)} |\Gamma(-3/4 + iv)| \left(T \sum_{n>T} \frac{d(n)}{n^{7/2-2\sigma}} + \sum_{n>T} \frac{d(n)}{n^{5/2-2\sigma}} \right)^{\frac{1}{2}} dv \right)^2 \\
&\ll_{\sigma} T^{\frac{5}{2}-2\sigma} \left(\left(T^{2\sigma-\frac{3}{2}} \log^3(T) \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} |\Gamma(-3/4 + iv)| dv \right)^2 \\
&\ll T \log^3(T),
\end{aligned}$$

where we used the estimate of $\Gamma(s)$ in Theorem 1.2 to see that the last integral is finite. Deducing the estimate for $J_6(s)$ is identical, but we assume that $1/4 < \sigma < 3/4$ to avoid the case $a = -1$ in Lemma 4.3.

We conclude that we have obtained the correct estimate for the third term on the right-hand side of (4.9). As a consequence, we immediately obtain the same desired estimate for the fourth term by using the Cauchy-Schwarz inequality.

For the second term, we write

$$\int_{T/2}^T J_1^2(1/2 + it) dt = \frac{1}{i} \int_{1/2+iT/2}^{1/2+iT} J_1^2(s) ds$$

and change the path of integration to avoid the case $a = -1$ in Lemma 4.3. Employing the usual estimation techniques yields

$$\begin{aligned}
\int_{T/2}^T J_1^2(1/2 + it) dt &= \frac{1}{i} \left(\int_{1/2+iT/2}^{1/4+iT/2} J_1^2(s) ds + \int_{1/4+iT/2}^{1/4+iT} J_1^2(s) ds + \int_{1/4+iT}^{1/2+iT} J_1^2(s) ds \right) \\
&\ll \int_{1/4}^{1/2} T^{2\sigma-1} \left(\sum_{n \leq T} \frac{d(n)}{n^{\sigma}} \right)^2 d\sigma + \int_{T/2}^T T^{-\frac{1}{2}} \left| \sum_{n \leq T} \frac{d(n)}{n^{1/4+it}} \right|^2 dt \\
&\ll \int_{1/4}^{1/2} T^{2\sigma-1} T^{2-2\sigma} \log^2(T) d\sigma + T^{-\frac{1}{2}} \left(T \sum_{n \leq T} \frac{d^2(n)}{n^{1/2}} + \sum_{n \leq T} d^2(n) n^{1/2} \right) \\
&\ll T \log^3(T).
\end{aligned}$$

To estimate the sum in the first integral, we used the estimate $\sum_{n \leq x} d(n) \ll x \log(x)$ that we achieved in the beginning of the proof of Lemma 4.3, together with partial summation. The same estimate for $\int_{T/2}^T J_2^2(1/2 + it) dt$ can similarly be achieved.

For the final term to be estimated in (4.9), we again change the path of integration, this time to a line segment with real part $3/8$. We also use the Cauchy-Schwarz inequality in every integral:

$$\begin{aligned}
&\int_{T/2}^T J_l(1/2 + it) J_k(1/2 + it) dt \\
&\ll \left(\int_{3/8}^{1/2} |J_l(\sigma + iT/2)|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{3/8}^{1/2} |J_k(\sigma + iT/2)|^2 d\sigma \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{T/2}^T |J_l(3/8 + it)|^2 dt \right)^{\frac{1}{2}} \left(\int_{T/2}^T |J_k(3/8 + it)|^2 dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$+ \left(\int_{3/8}^{1/2} |J_l(\sigma + iT)|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{3/8}^{1/2} |J_k(\sigma + iT)|^2 d\sigma \right)^{\frac{1}{2}}$$

with $l = 1, 2$ and $k = 3, 4, 5, 6$. It remains to estimate all of these integrals by $T \log^3(T)$.

The estimate of the vertical and horizontal parts with J_l is entirely the same as in the estimate of $\int_{T/2}^T J_1^2(1/2 + it) dt$. The estimate of the vertical part of J_k is already covered in the estimates for the integrals $\int_{T/2}^T |J_k(\sigma + it)|^2 dt$ above. The estimates of the horizontal parts of J_3 and J_4 are also the same as for J_1 , but where we also use the bounds $e^{-x} \ll 1/x^2$ and $|e^{-x} - 1| \leq 2$ respectively.

Therefore, the final estimate that we require concerns the horizontal parts of J_5 and J_6 . This is easily checked:

$$\begin{aligned} & \int_{3/8}^{1/2} |J_5(\sigma + iT)|^2 d\sigma \\ & \ll \int_{3/8}^{1/2} T^{2\sigma-1} \left(\int_{-\log^2(T)}^{\log^2(T)} T^{\frac{5}{2}-2\sigma} \sum_{n>T} \frac{d(n)}{n^{7/4-\sigma}} |\Gamma(-3/4 + iv)| T^{-\frac{3}{4}} dv \right)^2 d\sigma \\ & \ll \int_{3/8}^{1/2} T^{\frac{5}{2}-2\sigma} (T^{\sigma-3/4} \log(T))^2 d\sigma \ll T \log^2(T), \end{aligned}$$

and the estimate for J_6 or the horizontal part at height $T/2$ is the same. $\square$

This theorem is a classical result due to Ingham [20] from 1926. In 1977, Heath-Brown [16] improved the asymptotic estimate significantly to

$$\int_0^T |\zeta(1/2 + it)|^4 dt = \frac{1}{2\pi^2} T \log^4(T) + \sum_{k=0}^3 a_k T \log^{3-k}(T) + O_\varepsilon(T^{\frac{7}{8}+\varepsilon}),$$

where $a_0, \dots, a_3$ are real numbers. This is, as far as the author's knowledge goes, the best estimate to date.

4.3. Even power moments

Estimating the second and fourth moments is part of a more general study, namely that of even power moments

$$I_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt,$$

where k is a positive integer. In particular, these moments seem to be the most accessible to prove asymptotic formulas for. This is due to the identity

$$|\zeta(1/2 + it)|^{2k} = \zeta^k(1/2 + it) \overline{\zeta^k(1/2 + it)} = \zeta^k(1/2 + it) \zeta^k(1/2 - it),$$

which was exploited in the proof of both moment estimates above.

In the previous two theorems, we have already shown that

$$I_1(T) \sim T \log(T) \quad \text{and} \quad I_2(T) \sim \frac{1}{2\pi^2} T \log^4(T).$$

However, no unconditional asymptotic formulas have yet been proven for $k \geq 3$. Under certain conjectures, it has been shown in [28] and [29] that, respectively,

$$I_3(T) \sim a_3 T \log^9(T) \quad \text{and} \quad I_4(T) \sim a_4 T \log^{16}(T),$$

where

$$a_k = \left(\prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \right) \left(\prod_p \left(1 - \frac{1}{p} \right)^{k^2} \sum_{m=0}^{\infty} \binom{m+k-1}{k-1}^2 p^{-m} \right).$$

For the eighth moment, one of these underlying assumptions is the Riemann Hypothesis.

These four asymptotic formulas are consistent with a larger conjecture by Keating and Snaith [24] in 1998. Based on a random matrix model, they predicted that

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt \sim a_k T \log^{k^2}(T). \quad (4.10)$$

Indeed, it is trivial to calculate that $a_1 = 1$, and using the computations in the proof of Lemma 4.2, we find for $k = 2$ that

$$a_2 = \frac{1}{12} \prod_p \left(1 - \frac{1}{p} \right)^4 \sum_{m=0}^{\infty} (m+1)^2 p^{-m} = \frac{1}{12} \prod_p \left(1 - \frac{1}{p^2} \right) = \frac{1}{2\pi^2}.$$

An even stronger conjecture (see e.g. [10, Section 1.3]) suggests that there exists a polynomial $P_k(x)$ of degree k^2 such that

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt = T P_k(\log(T)) + O_{\varepsilon}(T^{1/2+\varepsilon})$$

for any $\varepsilon > 0$.

4.4. Real power moments

From now on, we will consider moments of an arbitrary fixed power $A \geq 0$. In particular, we will study estimates of the form

$$\int_0^T |\zeta(1/2 + it)|^A dt \ll_M T^M. \quad (4.11)$$

These integrals appear naturally in the zero-detection method of Chapter 5, namely when estimating the number of well-spaced class-II zeros. Therefore, it is particularly interesting to study how small the exponent M can be such that (4.11) holds. Accordingly, we define $M(A)$ to be the infimum of all such $M \in \mathbb{R}$. One aims to evaluate this function in every $A \geq 0$. This was carried out independently by the author in the range $0 \leq A \leq 4$. For larger values of A , the problem remains open, but upper bounds are proven (see [39, Section 2.1] for the current best results). After obtaining an equivalent estimate to (4.11), we will prove the current best bound in the interval $4 \leq A \leq 12$. We end this section by discussing the conjectured value of $M(A)$, and its equivalence with the Lindelöf hypothesis.

4.4.1. The value of $M(A)$ for exponents $0 \leq A \leq 4$

Using Hölder's inequality, we find that for all fixed $B > A > 0$ and $\varepsilon > 0$,

$$\int_0^T |\zeta(1/2 + it)|^A dt \leq \left(\int_0^T |\zeta(1/2 + it)|^B dt \right)^{\frac{A}{B}} \left(\int_0^T 1 dt \right)^{1 - \frac{A}{B}} \ll T^{\frac{A}{B}(M(B) + \varepsilon) + 1 - \frac{A}{B}},$$

and hence

$$M(A) \leq \frac{A}{B} M(B) + 1 - \frac{A}{B}. \quad (4.12)$$

Trivially, $M(0) = 1$, and from the estimates of the second and fourth moments, we know that $M(2) = M(4) = 1$.

First, we remark that $M(A) \geq 1$ for all $A \geq 0$. In [17], Heath-Brown showed that the lower bound

$$\int_0^T |\zeta(1/2 + it)|^A dt \gg T \log^{A^2/4}(T) \quad (4.13)$$

holds for all rational $A \geq 0$, proving $M(A) \geq 1$ in this case. Note that this is the lower bound that we expect from (4.10) if A is an even integer. By selecting any real number $B > 0$ and some positive rational number $A \leq B$ in (4.12), we see that $M(B) \geq 1$ for all real, non-negative B . It is proven that, under the Riemann Hypothesis, the lower bound (4.13) holds for any real exponent $A \geq 0$ [31].

The Riemann Hypothesis also provides Soundararajan's [34] upper bound

$$\int_0^T |\zeta(1/2 + it)|^A dt \ll_{\varepsilon} T \log^{A^2/4 + \varepsilon}(T)$$

for $A \geq 2$, which is nearly as good as what we expect from (4.10). In addition, the right bound has already been proven under the Riemann Hypothesis in the interval $0 \leq A \leq 24/11$, see [17, 30].

In order to obtain an unconditional upper bound, we can apply the definition of the growth exponent $\mu(\sigma)$, which implies the estimate $\zeta(1/2 + it) \ll_{\varepsilon} |t|^{\mu(1/2) + \varepsilon}$. This provides us with the trivial bound

$$\int_0^T |\zeta(1/2 + it)|^A dt \ll_{\varepsilon} T^{1 + A\mu(1/2) + A\varepsilon},$$

which implies the linear bound $M(A) \leq 1 + A\mu(1/2)$. By Theorem 1.10, we can plug in $\mu(1/2) = 13/84$.

Now, taking $B = 4$ in (4.12) shows us that $M(A) = 1$ for $0 \leq A \leq 4$. This is not unique to the value $B = 4$. In fact, for any $B \geq 0$ with $M(B) = 1$, the same argument shows that $M(A) = 1$ for $0 \leq A \leq B$.

4.4.2. An equivalent estimate

An important motivation for studying moment estimates is the equivalence with estimates for the sum

$$\sum_{r \leq R} |\zeta(1/2 + it_r)|^A,$$

where $t_1, \dots, t_R \in [T, 2T]$ satisfy the spacing condition $|t_r - t_s| \geq \log^C(T)$ if $r \neq s$. Here, $C \geq 0$ is a fixed constant. This type of sum naturally arises in the popular zero detection method discussed in Chapter 5, and motivates the study of moment estimates in this context. Moreover, it will help us prove an upper bound for $M(A)$ in the range $4 \leq A \leq 12$. We explain this equivalence with two propositions, based on the introduction of [22, Chapter 8].

Proposition 4.5. Let $A, M, C \geq 0$. Suppose that there is some $T_0 \geq 2$ such that for $T \geq T_0$ and $t_1, \dots, t_R \in [T, 2T]$ with $|t_r - t_s| \geq \log^C(T)$ if $r \neq s$, we have

$$\sum_{r \leq R} |\zeta(1/2 + it_r)|^A \ll T^M. \quad (4.14)$$

Then $M(A) \leq M$.

Proof. For $m = 1, \dots, \lceil T \log^C T \rceil$, let $I_m = [2T - m \log^C(T), 2T - (m-1) \log^C(T)] \cap [T, 2T]$ and take $t_m \in I_m$ such that $|\zeta(1/2 + it_m)|$ is maximal. Then, for any $\varepsilon > 0$, we find

$$\begin{aligned} & \int_T^{2T} |\zeta(1/2 + it)|^A dt \\ & \leq 2 \log^C(T) \left(\sum_{2m \leq \lceil T \log^C T \rceil} |\zeta(1/2 + it_{2m})|^A + \sum_{2m+1 \leq \lceil T \log^C T \rceil} |\zeta(1/2 + it_{2m+1})|^A \right) \\ & \ll_\varepsilon T^{M+\varepsilon}, \end{aligned}$$

since both sums satisfy the conditions from the proposition. Therefore,

$$\begin{aligned} \int_0^T |\zeta(1/2 + it)|^A dt & \leq \int_0^{2T_0} |\zeta(1/2 + it)|^A dt + \sum_{T/2^{n+1} \geq T_0} \int_{T/2^{n+1}}^{T/2^n} |\zeta(1/2 + it)|^A dt \\ & \ll_\varepsilon T^M + \sum_{n=0}^{\infty} \frac{T^{M+\varepsilon}}{(2^{M+\varepsilon})^n} \\ & \ll_\varepsilon T^{M+\varepsilon}. \end{aligned}$$

Hence, $M(A) \leq M + \varepsilon$ for all $\varepsilon > 0$ and we can conclude. $\square$

Proposition 4.6. Let $A \geq 0$ and fix any $C \geq 0$ and $\varepsilon > 0$. Then, for $T \geq 2$ and $t_1, \dots, t_R \in [T, 2T]$ with $|t_r - t_s| \geq \log^C(T)$ if $r \neq s$, we have

$$\sum_{r \leq R} |\zeta(1/2 + it_r)|^A \ll T^{M(A)+\varepsilon}.$$

Proof. **Case I: $A \geq 1$** We use (1.12) to estimate each term by an integral over a relatively small interval. This yields

$$\sum_{r \leq R} |\zeta(1/2 + it_r)|^A \ll T^{\frac{\varepsilon}{2}} \left(R + \sum_{r \leq R} \int_{-\log^2(T)}^{\log^2(T)} |\zeta(1/2 + it_r + iv)|^A dv \right).$$

Note that by the spacing condition on the t_r , we have $R \leq T \log^{-C}(T) + 1$. This condition also helps us control the overlap in the intervals over which we integrate in the sum. If $C \geq 2$, there are no more than two other t_j whose interval overlaps with that of t_r . Else, there are no more than $2\lfloor 2 \log^{2-C}(T) \rfloor$. Consequently,

$$\begin{aligned} \sum_{r \leq R} |\zeta(1/2 + it_r)|^A &\ll T^{\frac{\varepsilon}{2}} \left(T \log^{-C}(T) + (\log(T))^{\max\{0, 2-C\}} \int_{T-\log^2(T)}^{2T+\log^2(T)} |\zeta(1/2 + it)|^A dt \right) \\ &\ll T^{M(A)+\varepsilon}. \end{aligned}$$

Case II: $0 < A < 1$ Use Hölder's inequality, the case $A = 1$ shown above, and the fact that $M(1) = 1 = M(A)$. Also estimate R as in the previous case. This implies

$$\sum_{r \leq R} |\zeta(1/2 + it_r)|^A \leq \left(\sum_{r \leq R} |\zeta(1/2 + it_r)| \right)^A R^{1-A} \ll T^{A(M(1)+\varepsilon/A)+1-A} = T^{M(A)+\varepsilon}.$$

Case III: $A = 0$ This is trivial once we estimate R as above, since $M(0) = 1$. $\square$

Remark 4.7. We can extend this proposition to the same statement, but where we consider $t_1, \dots, t_R \in [0, T]$ instead. Indeed, let $k(T)$ denote the minimal integer such that $2^{-k(T)-1}T < 2$. Then, applying Proposition 4.6 to $T/2, T/4, \dots, T/2^{k(T)}$ and summing the results yields the desired estimate, as we can bound the values in all $t_r \in [0, 2]$ by a constant and $M(A) \geq 1$.

We conclude from these two propositions that once $A, C \geq 0$ are fixed, $M(A)$ is also the infimum of all $M \geq 1$ such that

$$\sum_{r \leq R} |\zeta(1/2 + it_r)|^A \ll T^M$$

for $T \geq 2$ and $t_1, \dots, t_R \in [T, 2T]$ with $|t_r - t_s| \geq \log^C(T)$ if $r \neq s$.

4.4.3. The current record for exponents $4 \leq A \leq 12$

In the following theorem, we prove the best known upper bound for $M(A)$ to date in the interval $4 \leq A \leq 12$. The bound for $A = 12$ will be used to obtain one of the zero-density estimates in Chapter 5. We follow the proof of [22, Thm. 8.3], which relies on a third estimate equivalent to (4.11), explained in [22, pp. 201].

Theorem 4.8. For $4 \leq A \leq 12$, we have

$$M(A) \leq 1 + \frac{A-4}{8}.$$

Proof. We use the equivalent formulation of moment estimates established in the previous section. Fix $C \geq 0$, $4 \leq A \leq 12$, and $\varepsilon > 0$. Since $|\zeta(1/2 + it)| \ll |t|^{13/84+\varepsilon}$ (see Theorem 1.10) and $13/84 < 1/6$, there exists some $T_0 \geq 2$ such that $|\zeta(1/2 + it)| < (t/2)^{1/6}$ if $t \geq T_0$. Now take $T_0 \leq T \leq t_1 \leq \dots \leq t_R \leq 2T$ with $|t_r - t_s| \geq \log^C(T)$ if $r \neq s$.

We consider a dyadic subdivision of the interval $[1, T^{1/6}]$. Regroup the terms in (4.14) depending on the value of the integer k for which we have $2^{-k-1}T^{1/6} < |\zeta(\sigma + it_r)| \leq 2^{-k}T^{1/6}$. Denote the set of such $r \leq R$ by R_k . Since $|\zeta(1/2 + it_r)| < (t_r/2)^{1/6} \leq T^{1/6}$ for all $r \leq R$, this yields

$$\begin{aligned} \sum_{r \leq R} |\zeta(1/2 + it_r)|^A &\leq \sum_{\substack{r \leq R \\ |\zeta(1/2 + it_r)| \leq 1}} |\zeta(1/2 + it_r)|^A + \sum_{\substack{k \geq 0 \\ 2^k < T^{1/6}}} \sum_{r \in R_k} |\zeta(1/2 + it_r)|^A \\ &\leq R + \sum_{\substack{k \geq 0 \\ 2^k < T^{1/6}}} 2^{-Ak} T^{A/6} |R_k|. \end{aligned} \quad (4.15)$$

We continue by bounding $|R_k|$ using our results on large values of the Riemann zeta function. Set $\alpha = (12 - A)/8$ and note that $0 \leq \alpha \leq 1$. Then, apply (3.7) and (3.8) with $V = 2^{-k-1}T^{1/6}$ to see

$$\begin{aligned} |R_k| &= |R_k|^\alpha |R_k|^{1-\alpha} \ll (TV^{-4} \log^8(T))^\alpha (T^2 V^{-12} \log^{16}(T))^{1-\alpha} \\ &= T^{2-\alpha} V^{-(12-8\alpha)} (\log(T))^{16-8\alpha} \ll T^{(4+A)/8+\varepsilon} \cdot 2^{A(k+1)} T^{-A/6}. \end{aligned}$$

We substitute this estimate into (4.15) and use the well-spaced condition to get

$$\begin{aligned} \sum_{r \leq R} |\zeta(1/2 + it_r)|^A &\ll T + \sum_{\substack{k \geq 0 \\ 2^k < T^{1/6}}} 2^A T^{(4+A)/8+\varepsilon} \\ &\ll T + 2^A T^{(4+A)/8+\varepsilon} \log(T) \\ &\ll T^{(4+A)/8+2\varepsilon}. \end{aligned}$$

Hence, Proposition 4.5 provides the inequality $M(A) \leq (4 + A)/8 = 1 + (A - 4)/8$. $\square$

4.4.4. The conjectured value of $M(A)$ and its relation to the Lindelöf hypothesis

It has been conjectured that $M(A) = 1$ for all $A \geq 0$. In fact, this is equivalent to the Lindelöf hypothesis, which asserts that $\zeta(1/2 + it) \ll_\varepsilon |t|^\varepsilon$ for all $\varepsilon > 0$. The implication from the Lindelöf hypothesis to $M(A) = 1$ can be trivially achieved by substituting this estimate into the moment integral. For the converse implication, we use Corollary 1.12, which estimates $|\zeta(1/2 + it)|^A$ by a moment integral. Suppose that $M(A) = 1$ for all $A \geq 0$. Fix any $A \geq 1$, then the corollary and our assumption imply

$$|\zeta(1/2 + iT)|^A \ll \log^4(T) \left(1 + \int_{T-\log^2(T)}^{T+\log^2(T)} |\zeta(1/2 + it)|^A dt \right) \ll T^2.$$

Therefore, $|\zeta(1/2 + iT)| \ll T^{\frac{2}{A}}$, so choosing $A \geq 2/\varepsilon$ yields the Lindelöf hypothesis.

It is interesting to remark that proving the Lindelöf hypothesis only required $M(A_n) = 1$ for a sequence of positive real numbers $(A_n)_{n \in \mathbb{N}}$ with $A_n \rightarrow \infty$. Consequently, this condition is enough to establish $M(A) = 1$ for all $A \geq 0$. This can also be seen by plugging in $B = A_n$ into (4.12), which shows that $M(A) = 1$ for all $A \leq A_n$. In particular, the conjectured asymptotic formulas for the even power moments in (4.10) imply this.

Chapter 5

Zero-density estimates

With the tools developed in earlier chapters, we are now ready for the main topic of this thesis. Define $N(\sigma, T)$ as the number of non-trivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function in the rectangle given by $\sigma \leq \beta \leq 1$ and $|\gamma| \leq T$, counted with multiplicity. The goal of zero-density estimates is to estimate this number. By the Riemann-von Mangoldt formula (1.7) and the symmetry of the non-trivial zeros with respect to the line $\Re(s) = 1/2$, we have $T \log(T) \ll N(\sigma, T) \ll T \log(T)$ if $\sigma \leq 1/2$. Therefore, we restrict our attention to the range $1/2 \leq \sigma \leq 1$.

We consider two types of zero-density estimates, which we will dub crude and refined estimates. A crude estimate is of the form

$$N(\sigma, T) \ll_{\varepsilon} T^{A(\sigma)(1-\sigma)+\varepsilon}, \quad (5.1)$$

for any $\varepsilon > 0$, where $A(\sigma)$ is a function of σ . On the other hand, a refined estimate is a bound of the form

$$N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)} \log^B(T), \quad (5.2)$$

where the *crude* factor T^{ε} is replaced by the more *refined* factor $\log^B(T)$ for some $B > 0$.

In particular, we give special attention to an estimate of the form

$$N(\sigma, T) \ll T^{c(1-\sigma)+\varepsilon}$$

in the whole interval $1/2 \leq \sigma \leq 1$, where $c > 0$ is a fixed constant. In this chapter, we obtain this estimate for $c = 12/5$. Such a result is one of the two key results in the proof that derives estimates for primes in short intervals in the next chapter. In particular, proving the constant $c = 2$ yields an estimate for this problem of comparable sharpness to the one implied by the Riemann Hypothesis. This statement is referred to as a density hypothesis, which is known to hold in $\sigma \geq 25/32 = 0.78125$ [6].

We begin by describing the zero-detection method, which is the standard method by now, but still offers great flexibility. This will happen in two parts. In the first part, we distinguish two types of non-trivial zeros, called the class-I and class-II zeros. The problem then reduces to estimating the number of zeros in both classes. Before we complete the full method, we pause to first establish a refined zero-density estimate. Such estimates require a more careful analysis that is not offered by the remainder of

the method, which is primarily suited to obtain crude estimates. The second part will complete the zero-detection method, after which it will be applied to obtain several crude zero-density estimates. This includes a new result that improves the current best zero-density estimates (see [36, Table 11.1]) in an interval of length 0.009.

Throughout this chapter, the constant in our estimates will only depend on ε , which can attain any sufficiently small positive value. The exact value of ε is allowed to increase throughout our methods, but only by a constant factor. Finally, we always assume T to be sufficiently large for the deductions we make, which may depend on the value of ε .

5.1. The zero-detection method, Part I

We first introduce two parameter functions $X = X(\sigma, T)$ and $Y = Y(\sigma, T)$, whose values will be chosen later depending on σ and T . This allows us to choose the optimal functions at a later stage. We impose the conditions $T^\varepsilon \leq X \leq Y \leq T^C$. Here, we have fixed a sufficiently large value $C > 0$.

We begin by deriving an equation that will allow us to classify the non-trivial zeros into two classes. The starting point is the Mellin inversion formula (4.5) once more. We apply this for $x = n/Y$ and $c = 2$ to obtain

$$e^{-n/Y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w) Y^w n^{-w} dw. \quad (5.3)$$

We introduce the sequence

$$a(n) = \sum_{d|n, d \leq X} \mu(d),$$

which is the Dirichlet convolution of the sequences 1 and $\mu(n)1_{[0, X]}(n)$, where μ is the Möbius function and 1_A is the indicator function of the set A . The latter sequence has Dirichlet series $M_X(s) = \sum_{n \leq X} \mu(n)n^{-s}$, so the Dirichlet series of $a(n)$ is $\zeta(s)M_X(s)$, absolutely converging for $\sigma > 1$. Recall the identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n > 1, \end{cases}$$

see e.g. [37, Thm. 2.8], which implies that $a(n) = 0$ if $1 < n \leq X$. Finally, it will also be useful to note that $|a(n)| \leq d(n) \ll n^\varepsilon$. This is a well-known estimate, see e.g. [22, Equation (1.70)].

Suppose now that $\rho = \beta + i\gamma$ is a non-trivial zero of the Riemann zeta function with $\sigma \leq \beta$ and $|\gamma| \leq T$. Using (5.3) and the Dominated Convergence Theorem with Stirling's formula in Theorem 1.2, we get

$$e^{-1/Y} + \sum_{n > X} a(n)n^{-\rho}e^{-n/Y} = \sum_{n=1}^{\infty} a(n)n^{-\rho}e^{-n/Y}$$

$$= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(\rho+w) M_X(\rho+w) Y^w \Gamma(w) dw.$$

Next, we move the line of integration to $\Re(s) = 1/2 - \beta$. We can bound the horizontal parts with the estimates for the Gamma function and the zeta function in Theorem 1.2 and Theorem 1.9 respectively, implying that they tend to zero. Note that the pole of $\Gamma(w)$ in $w = 0$ is canceled by the zero of $\zeta(\rho+w)$, so the only pole we come across is in $w = 1 - \rho$. Again applying Theorem 1.2, it is easily seen that this residue is $O(e^{-\log^2(T)})$ if $|\gamma| \geq \log^2(T)$. In this case, we therefore get

$$\begin{aligned} e^{-1/Y} + \sum_{n>X} a(n) n^{-\rho} e^{-n/Y} &= \frac{1}{2\pi i} \int_{1/2-\beta-i\infty}^{1/2-\beta+i\infty} \zeta(\rho+w) M_X(\rho+w) Y^w \Gamma(w) dw + O(e^{-\log^2(T)}) \\ &= \frac{1}{2\pi} \int_{-\log^2(T)}^{\log^2(T)} \zeta\left(\frac{1}{2} + i\gamma + iv\right) M_X\left(\frac{1}{2} + i\gamma + iv\right) Y^{\frac{1}{2}-\beta+iv} \Gamma\left(\frac{1}{2} - \beta + iv\right) dv \\ &\quad + O(e^{-\log^2(T)}), \end{aligned} \tag{5.4}$$

where we applied the estimates for the Gamma function and zeta function once more.

We now estimate the tail of the infinite sum. Here we introduce the function

$$\alpha = \alpha(\sigma, T) = \left\lceil \frac{1}{\log(2)} \log\left(\frac{Y \log^2(Y)}{X}\right) \right\rceil \ll \log(Y),$$

such that $2^{\alpha-1}X < Y \log^2(Y) \leq 2^\alpha X$. This will allow us to easily use a dyadic subdivision of the interval $(X, 2^\alpha X] \supseteq (X, Y \log^2(Y)]$ later. We estimate

$$\begin{aligned} \sum_{n>2^\alpha X} a(n) n^{-\rho} e^{-n/Y} &\ll \sum_{n>Y \log^2(Y)} n^{\varepsilon-\beta} e^{-n/Y} \leq \int_{Y \log^2(Y)-1}^{\infty} e^{-t/Y} dt \\ &\ll Y e^{-\log^2(Y)} \ll e^{-\frac{\varepsilon^2}{2} \log^2(T)}. \end{aligned}$$

Finally, we estimate

$$e^{-1/Y} = 1 + O\left(\frac{1}{Y}\right) = 1 + O\left(\frac{1}{T^\varepsilon}\right).$$

We conclude by plugging these into our equation (5.4) to deduce

$$\begin{aligned} 1 + O\left(\frac{1}{T^\varepsilon}\right) + \sum_{X < n \leq 2^\alpha X} a(n) n^{-\rho} e^{-n/Y} &= \frac{1}{2\pi} \int_{-\log^2(T)}^{\log^2(T)} \zeta\left(\frac{1}{2} + i\gamma + iv\right) M_X\left(\frac{1}{2} + i\gamma + iv\right) Y^{\frac{1}{2}-\beta+iv} \Gamma\left(\frac{1}{2} - \beta + iv\right) dv \end{aligned}$$

if $|\gamma| \geq \log^2(T)$.

This equation is useful because it asserts that, if T is large enough, one of the following three conditions must hold for any zero counted in $N(\sigma, T)$:

$$|\gamma| \leq \log^2(T), \quad (5.5)$$

$$\left| \sum_{X < n \leq 2^\alpha X} a(n) n^{-\rho} e^{-n/Y} \right| \geq \frac{1}{3} \quad (5.6)$$

$$\left| \frac{1}{2\pi i} \int_{-\log^2(T)}^{\log^2(T)} \zeta \left(\frac{1}{2} + i\gamma + iv \right) M_X \left(\frac{1}{2} + i\gamma + iv \right) Y^{\frac{1}{2} - \beta + iv} \Gamma \left(\frac{1}{2} - \beta + iv \right) dw \right| \geq \frac{1}{3}. \quad (5.7)$$

The number of zeros for which (5.5) holds is $O(\log^3(T))$ by the Riemann-von Mangoldt formula (1.7). The zeros satisfying (5.6) and (5.7), but not (5.5), are respectively called class-I and class-II zeros.

Because it will later be useful to only consider zeros whose imaginary parts are sufficiently separated, we construct a well-spaced set R_1 of class-I zeros as follows. We start with an empty set R_1 and add a new class-I zero in every step. In the first step, we add the class-I zero ρ_n for which $\gamma > \log^2(T)$ is minimal. For this purpose, we may assume that such a zero exists for T sufficiently large, since otherwise we trivially have $N(\sigma, T) = 0$ for all $T \geq 0$. For any $n \geq 2$, the n -th step consists of picking the class-I zero ρ_n with minimal $\gamma_n > \gamma_{n-1} + 5 \log^2(T)$. We end once there are no such zeros anymore with $\gamma_n \leq T$. This way, we obtain a set R_1 of class-I zeros for which $|\gamma - \gamma'| \geq 5 \log^2(T)$ if $\rho \neq \rho' \in R_1$.

Since we do not account for all class-I zeros, we require an additional factor besides $|R_1|$ to estimate the number of these zeros. The Riemann-von Mangoldt formula (1.7) implies that

$$N(A + 5 \log^2(T)) - N(A) \ll A \log \left(1 + \frac{5 \log^2(T)}{A} \right) + \log^3(T)$$

if $\log^2(T) \leq A \leq T$. In case $\log^2(T) \leq A \leq 6 \log^2(T)$, we obtain $N(A + 2 \log^2(T)) - N(A) \ll \log^3(T)$. If instead $6 \log^2(T) < A \leq T$, we have the same bound by using the estimate $\log(1+x) \ll x$ for $|x| \leq 5/6$, which follows from the Taylor series of $\log(1+s)$ in $s=0$. In particular, the total number of class-I zeros is $\ll |R_1| \log^3(T)$.

We follow the same procedure for the class-II zeros to construct a similarly well-spaced set R_2 , such that the number of these zeros is $\ll |R_2| \log^3(T)$. In summary, we have now established the bound

$$N(\sigma, T) \ll (1 + |R_1| + |R_2|) \log^3(T). \quad (5.8)$$

It thus remains to estimate $|R_1|$ and $|R_2|$.

This ends the explanation of the zero-detection method for sharp estimates of the form (5.2). The method continues for crude estimates of the form (5.1), and this will be discussed in section 5.3.

5.2. A sharp estimate for $1/2 \leq \sigma \leq 3/4$

With the zero-detection method from the previous section, we are now ready to prove a first zero-density estimate. This section is dedicated to proving the following theorem, whose crude variant was first shown by A.E. Ingham in 1940 [21]. We follow the proof in [22, pp. 276].

Theorem 5.1. Uniformly for $1/2 \leq \sigma \leq 3/4$ and $T \geq 2$, we have

$$N(\sigma, T) \ll T^{3\frac{1-\sigma}{2-\sigma}} \log^{10}(T). \quad (5.9)$$

Proof. We start by bounding $|R_1|$. To this end, recall that the inequality (5.6) holds for all zeros ρ in R_1 . Since $\alpha \ll \log(Y)$, a dyadic subdivision of the sum and an application of the pigeonhole principle imply that there exists a non-negative integer $m \leq \alpha - 1$ such that

$$\left| \sum_{2^m X < n \leq 2^{m+1} X} a(n) n^{-\rho} e^{-n/Y} \right| \gg \frac{1}{\log(Y)} \quad (5.10)$$

for $\gg |R_1|/\log(Y)$ of those zeros. We denote $M = 2^m X$ and remark that $X \leq M \leq 2^{\alpha-1} X < Y \log^2(Y)$.

Let A be the set of all zeros for which (5.10) holds. We continue by summing this estimate over all zeros $\rho \in A$. Next, we square both sides and use the Cauchy-Schwarz inequality in order to apply the discrete form of the mean value theorem for Dirichlet polynomials in Theorem 2.5. We find

$$\frac{|A|^2}{\log^2(Y)} \ll \left(\sum_{\rho \in A} \left| \sum_{M < n \leq 2M} a(n) n^{-\rho} e^{-n/Y} \right| \right)^2 \leq |A| \sum_{\rho \in A} \left| \sum_{M < n \leq 2M} a(n) n^{-\rho} e^{-n/Y} \right|^2.$$

This enables us to invoke Theorem 2.5, establishing

$$\frac{|R_1|}{\log(Y)} \ll |A| \quad (5.11)$$

$$\begin{aligned} & \ll \log^2(Y) \sum_{\rho \in A} \left| \sum_{M < n \leq 2M} a(n) n^{-\beta-i\gamma} e^{-n/Y} \right|^2 \\ & \ll \log^2(T) \left(T \sum_{M < n \leq 2M} |a(n)|^2 n^{-2\beta} e^{-2n/Y} + \sum_{M < n \leq 2M} n |a(n)|^2 n^{-2\beta} e^{-2n/Y} \right) \log(M) \\ & \ll \log^3(T) \left(T \sum_{M < n \leq 2M} d^2(n) n^{-2\sigma} + \sum_{M < n \leq 2M} d^2(n) n^{1-2\sigma} \right) e^{-2M/Y}. \end{aligned} \quad (5.12)$$

We now wish to apply a similar estimate as in Lemma 4.3. However, we cannot directly apply this lemma, since the coefficient would depend on σ , and is not bounded in $1/2 \leq \sigma \leq 1$. Therefore, we present a different partial summation argument starting

from (4.4). In the partial summation formula, we use the sequence $d^2(n)1_{(x,\infty)}(n)$ and the function $f(x) = x^a$, where a can be any real number. The constant in the estimate will be independent of both x and a . We find

$$\begin{aligned} \sum_{x < n \leq 2x} d^2(n)n^a &= x^a \sum_{x < n \leq 2x} d^2(n) - \int_x^{2x} \left(\sum_{x < n \leq t} d^2(n) \right) at^{a-1} dt \\ &\ll \sum_{x < n \leq 2x} d^2(n) \left(x^a + \int_x^{2x} |at^{a-1}| dt \right) \\ &\ll (1 + 2^a) x^{1+a} \log^3(x). \end{aligned}$$

Plugging this into (5.12) yields

$$|R_1| \ll \log^7(T) (TM^{1-2\sigma} + M^{2-2\sigma}) e^{-2M/Y}.$$

By studying the derivative of the function $f(x) = x^{2-2\sigma} e^{-2x/Y}$ defined for $x \geq 0$, we see that its maximum is reached in $x = (1 - \sigma)Y$. Therefore

$$|R_1| \ll \log^7(T) (TX^{1-2\sigma} + Y^{2-2\sigma}).$$

Hence, we are now left to choose the optimal functions for X and Y . However, we will first bound $|R_2|$, as its estimate may also include X and Y .

For every zero ρ in R_2 , take $t_\rho \in [\gamma - \log^2(T), \gamma + \log^2(T)]$ such that $|\zeta(1/2 + it_\rho)M_X(1/2 + it_\rho)|$ is maximal. This allows us to infer from (5.7) that

$$|\zeta(1/2 + it_\rho)M_X(1/2 + it_\rho)|Y^{1/2-\sigma} \gg 1, \quad (5.13)$$

where we used Stirling's formula (1.5) to bound the integral over the Gamma function by an absolute constant. Note that, once T is sufficiently large, we have $0 \leq t_\rho \leq 2T$ and $|t_\rho - t_{\rho'}| \geq \log^2(2T)$ for $t_\rho \neq t_{\rho'}$.

We raise (5.13) to the power $4/3$ and sum over all zeros counted in R_2 , followed by an application of Hölder's inequality. This yields

$$\begin{aligned} |R_2| &\ll Y^{(2-4\sigma)/3} \sum_{\rho \in R_2} |\zeta(\tfrac{1}{2} + it_\rho)|^{4/3} |M_X(\tfrac{1}{2} + it_\rho)|^{4/3} \\ &\ll Y^{(2-4\sigma)/3} \left(\sum_{\rho \in R_2} |\zeta(\tfrac{1}{2} + it_\rho)|^4 \right)^{1/3} \left(\sum_{\rho \in R_2} |M_X(\tfrac{1}{2} + it_\rho)|^2 \right)^{2/3}. \end{aligned}$$

For the first sum, we use the fourth moment estimate (4.6) in combination with Proposition 4.6 in the form given by Remark 4.7. In fact, a more careful consideration of the proof of Proposition 4.6 for $A = 4$, using the sharper bound in Lemma 1.11, reveals that we can estimate this sum as $\ll T \log^5(T)$. In order to estimate the second sum, we again make

use of Theorem 2.5, followed by Euler's summation formula, and we impose the condition $X \leq T \log(T)$. This yields

$$\begin{aligned} |R_2| &\ll Y^{(2-4\sigma)/3} (T \log^5 T)^{1/3} \left(T \sum_{n \leq X} n^{-1} + \sum_{n \leq X} 1 \right)^{2/3} \log^{2/3}(T) \\ &\ll Y^{(2-4\sigma)/3} T \log^3(T). \end{aligned}$$

We now have an estimate for both $|R_1|$ and $|R_2|$. Substituting these into (5.8) gives

$$N(\sigma, T) \ll (Y^{2-2\sigma} + TX^{1-2\sigma} + TY^{(2-4\sigma)/3}) \log^{10}(T).$$

We are left with finding the optimal functions X and Y that minimizes expression. Since the exponent of Y in the first and third term are positive and negative respectively, we will find the optimal Y when both terms are equal. This gives $Y = T^{\frac{3}{4-2\sigma}}$. The exponent of X is negative, so we need to choose X as large as possible, given the imposed conditions, i.e. $X \leq \min\{T \log(T), Y\}$. Due to the other two terms, a logarithmic factor for X will be insignificant, so we pick $X = T$. Then $T^\varepsilon \leq X = T \leq Y \leq T^{3/2}$, hence all conditions on X and Y are met. When filling in X and Y , the final estimate becomes (5.9). $\square$

Remark 5.2. The above proof shows that the estimate actually holds in the bigger interval $1/2 \leq \sigma \leq 1$. However, we only consider a smaller interval, since a better bound will be proven in section 5.4 for $3/4 \leq \sigma \leq 1$. Together, they will provide an estimate similar to the density hypothesis, but for the constant $12/5$ instead of 2.

At present, this is still, up to a logarithmic factor, the best known zero-density estimate in $1/2 \leq \sigma \leq 7/10$ according to the ANTEDB [36, Table 11.1]. Improving it around $\sigma = 7/10$ would immediately lower the best known constant $c = 30/13$ in the estimate $N(\sigma, T) \ll T^{c(1-\sigma)+\varepsilon}$, uniformly over $1/2 \leq \sigma \leq 1$.

5.3. The zero-detection method, Part II

In this section, we continue the zero-detection method explained in section 5.1. From this point onward, we allow factors of the form T^ε , so only crude estimates can be achieved. By the end of this section, the estimate for the class-II zeros will essentially be completed (given that we have moment estimates), while the class-I zeros still require additional work.

5.3.1. The class-I zeros

As in the beginning of the proof of Theorem 5.1, we find some $M \in [X, Y \log^2(Y))$ such that

$$\left| \sum_{M < n \leq 2M} a(n) n^{-\rho} e^{-n/Y} \right| \gg \frac{1}{\log(Y)} \quad (5.14)$$

holds for $\gg |R_1|/\log(Y)$ of the well-spaced class-I zeros. Denote $\tilde{R}_1 \subseteq R_1$ for the subset of zeros for which (5.14) holds. Consider now the integer k for which $M^k \leq Y^2 \log^4(Y) <$

M^{k+1} and note that $2 \leq k \leq \frac{2 \log(Y \log^2(Y))}{\log(X)} \leq \frac{4C \log(T)}{\varepsilon \log(T)} \ll 1$ if T (and therefore Y) is large enough. The lower bound implies $Y^{4/3} \log^{8/3}(Y) < M^k \leq Y^2 \log^4(Y)$. Raise (5.14) to the power k to find

$$\left| \sum_{M^k < n \leq 2^k M^k} a'(n) n^{-\rho} \right| \gg \frac{1}{\log^k(Y)},$$

where

$$a'(n) = \sum_{\substack{M < n_1, \dots, n_k \leq 2M \\ n_1 \cdots n_k = n}} a(n_1) \cdots a(n_k) e^{-\frac{n_1 + \dots + n_k}{Y}} \ll d_{2k}(n) \ll n^\varepsilon,$$

see e.g. [22, Equation (1.70)].

Pick any subset $R \subseteq \tilde{R}_1$. Our aim is now to estimate $|R|$ in order to apply Huxley's subdivision trick later. However, we need to carefully consider which parameters depend on our choice of R . Moreover, the constant in the estimate must remain independent of R .

Since $k \ll 1$, A dyadic subdivision shows that there is some $N = N(R) \in [M^k, 2^k M^k)$, depending on R , such that

$$\left| \sum_{N < n \leq 2N} a'(n) n^{-\rho} \right| \gg \frac{1}{\log^k(Y)}$$

for $\gg |R|$ elements of R . Summing over these elements and using partial summation yields

$$\begin{aligned} |R| &\ll \log^k(Y) \sum_{\rho} \left| \sum_{N < n \leq 2N} a'(n) n^{-\rho} \right| \\ &\leq \log^k(Y) \sum_{\rho \in R} \left| (2N)^{-\beta} \sum_{N < n \leq 2N} a'(n) n^{-i\gamma} + \int_N^{2N} \beta u^{-\beta-1} \sum_{N < n \leq u} a'(n) n^{-i\gamma} du \right| \\ &\leq \log^k(Y) \left((2N)^{-\beta} + \int_N^{2N} |\beta u^{-\beta-1}| du \right) \max_{N \leq v \leq 2N} \sum_{\rho \in R} \left| \sum_{N < n \leq v} a'(n) n^{-i\gamma} \right| \\ &\leq \log^k(Y) N^{-\sigma} \max_{N \leq v \leq 2N} \sum_{\rho \in R} \left| \sum_{N < n \leq v} a'(n) n^{-i\gamma} \right|. \end{aligned}$$

Take $v \in [N, 2N]$ such that the above sum is maximal, then define $a''(n) = a'(n)$ if $N < n \leq v$ and $a''(n) = 0$ elsewhere. Consequently,

$$|R| \ll N^{\frac{\varepsilon}{2} - \sigma} \sum_{\rho \in R} \left| \sum_{N < n \leq 2N} a''(n) n^{-i\gamma} \right|.$$

Define $b(n) = a''(n)/(2N)^{\varepsilon/2}$; this sequence also depends on R . Use the fact that $a''(n) \ll n^{\varepsilon/3}$ to see that $|b(n)| \leq 1$ for sufficiently large T . We conclude with

$$|R| \ll N^{\varepsilon - \sigma} \sum_{\rho \in R} \left| \sum_{N < n \leq 2N} b(n) n^{-i\gamma} \right|. \quad (5.15)$$

Here, N and $b(n)$ still depend on R , and the implicit constant may depend on the value of ε .

This is where the general zero-detection method ends for the class-I zeros. At this point, we will apply the Halász-Montgomery inequality (see Lemma 5.3 below) in sections 5.4 and 5.5. However, we will do so in two different ways, which is why we end the discussion here.

5.3.2. The class-II zeros

To estimate $|R_2|$, we recall (5.7), which holds for every zero ρ in R_2 . This time, we estimate $M_X(1/2 + i\gamma + iv) \ll X^{1/2}$ with Euler's summation formula and pick $t_\rho \in [\gamma - \log^2(T), \gamma + \log^2(T)]$ such that $|\zeta(1/2 + it_\rho)|$ is maximal. Note that again, once T is sufficiently large, we have $0 \leq t_\rho \leq 2T$ and $|t_\rho - t_{\rho'}| \geq \log^2(2T)$ for $t_\rho \neq t_{\rho'}$. Similarly to before, the estimate (5.7) gives us

$$X^{1/2} Y^{\frac{1}{2}-\sigma} |\zeta(1/2 + it_\rho)| \gg 1$$

for every $\rho \in R_2$.

Now we have two options. The first option is to raise both sides by a fixed real power $A > 0$ to be chosen later, sum over all zeros in R_2 , and use our moment estimates via Proposition 4.6 in the form stated in Remark 4.7. The second option is to fix an exponent pair (κ, λ) , which will be determined later, and use the theory of large values via Theorem 3.8 with $V = Y^{\sigma-\frac{1}{2}} X^{-\frac{1}{2}}$. In both cases, the smallest estimate for $|R_2|$ is reached when X is as small as possible, and since the bound (5.15) for the class-I zeros does not depend on X , we can afford to pick such a value. Therefore, we already fix $X = T^\varepsilon$. We find

$$|R_2| \ll T^\varepsilon Y^{A(1/2-\sigma)} \sum_{\rho \in R_2} |\zeta(1/2 + it_\rho)|^A \ll T^{M(A)+\varepsilon} Y^{A(1/2-\sigma)} \quad (5.16)$$

and

$$|R_2| \ll T^\varepsilon \left(TY^{3-6\sigma} + T^{\frac{\kappa+\lambda}{\kappa}} Y^{(1-2\sigma)\frac{1+2\kappa+2\lambda}{\kappa}} \right). \quad (5.17)$$

It remains to choose the power A or the exponent pair (κ, λ) , which is something to be optimized once we have a bound for R_1 .

5.4. A crude estimate for $3/4 \leq \sigma \leq 1$

With the full zero-detection method from the previous section at hand, we now proceed to prove crude zero-density estimates. A first result is given in the theorem below, which is the main focus of this section. This result was established by M. N. Huxley in 1972 [19]. We will follow the argument in [22, Thm. 11.1].

As mentioned before, the first step will be to apply the Halász-Montgomery inequality. For the convenience of the reader, this result is mentioned below. A proof can be found in [27, Lemma 1.7].

Lemma 5.3 (Halász-Montgomery inequality). Suppose $\xi, \varphi_1, \dots, \varphi_m$ are elements of a complex vector space with inner product $\langle \cdot, \cdot \rangle$. Then

$$\sum_{k=1}^m |\langle \xi, \varphi_k \rangle| \leq \|\xi\| \left(\sum_{1 \leq k, l \leq m} |\langle \varphi_k, \varphi_l \rangle| \right)^{1/2}.$$

Theorem 5.4. Fix any $\varepsilon > 0$. Then, uniformly in $3/4 \leq \sigma \leq 1$ and $T \geq 1$, we have

$$N(\sigma, T) \ll T^{3\frac{1-\sigma}{3\sigma-1}+\varepsilon}. \quad (5.18)$$

Proof. The main difficulty is to estimate the contribution of the class-I zeros. For this, we continue with (5.15) and apply the previous lemma to $\xi(n) = b(n)$ and $\varphi_\rho(n) = n^{i\gamma}$, defined for $N < n \leq 2N$. Note that $\|\xi\|^2 \ll N$, so we find

$$|R|^2 \ll T^\varepsilon N^{1-2\sigma} \left(|R|N + \sum_{\rho \neq \rho' \in R} \left| \sum_{N < n \leq 2N} n^{i\gamma - i\gamma'} \right| \right).$$

The latter sum is an exponential sum, which means that we can employ the theory of Chapter 3.

We now choose R to be the zeros of $\tilde{R}_1$ with imaginary parts in an arbitrary subinterval of $[-T, T]$ with length not exceeding $T_0 \in [1, 2T]$. The goal is to bound $|R|$ and sum up the contributions of $\ll T/T_0$ intervals to estimate $|\tilde{R}_1|$. Here, T_0 is a parameter that will be chosen later.

We distinguish two cases. In case $|\gamma - \gamma'| \leq N$, we use Lemma 3.1 with $f(x) = \frac{\gamma - \gamma'}{2\pi} \log(x)$ and $\delta = (2\pi)^{-1}$. This yields

$$\sum_{N < n \leq 2N} n^{i\gamma - i\gamma'} \ll 1 + \int_N^{2N} x^{i\gamma - i\gamma'} dx \ll 1 + \frac{N}{|\gamma - \gamma'|}.$$

In the other case, i.e. when $|\gamma - \gamma'| > N$, we make use of the exponent pair $(1/2, 1/2)$ (see Proposition 3.5). In Definition 3.4, which introduced exponent pairs, let $f(x)$ be the same function as before and take $A = |\gamma - \gamma'|/N > 1$ and $B = h = N$. We find

$$\sum_{N < n \leq 2N} n^{i\gamma - i\gamma'} \ll \left(\frac{|\gamma - \gamma'|}{N} \right)^{1/2} N^{1/2} \ll T_0^{1/2}.$$

In both cases, we have

$$\sum_{N < n \leq 2N} n^{i\gamma - i\gamma'} \ll \frac{N}{|\gamma - \gamma'|} + T_0^{1/2},$$

so

$$|R|^2 \ll T^\varepsilon N^{1-2\sigma} \left(|R|N + T_0^{1/2}|R|^2 + N \sum_{\rho \neq \rho' \in R} \frac{1}{|\gamma - \gamma'|} \right). \quad (5.19)$$

The only variable here that still depends on our choice of R is N . However, since $M^k \leq N < 2^k M$ and $k \ll 1$, we have $N(\tilde{R}_1) \ll N(R) \ll N(\tilde{R}_1)$. Therefore, the above estimate still holds if we replace $N(R)$ with $N(\tilde{R}_1)$. From now on in this proof, whenever we mention N , we mean $N(\tilde{R}_1)$.

Since $|\gamma - \gamma'| \geq 5 \log^2(T)$ if $\rho \neq \rho'$, we can estimate the latter sum by

$$\sum_{\rho \in R} \sum_{\substack{\rho' \in R \\ \rho' \neq \rho}} \frac{1}{|\gamma - \gamma'|} \leq \sum_{\rho \in R} \frac{2}{5 \log^2(T)} \sum_{n \leq T} \frac{1}{n} \ll |R|.$$

We also aim to move the second term in the right-hand side of (5.19) to the left-hand side, but for this, we require its coefficient $T^\varepsilon N^{1-2\sigma} T_0^{1/2}$ to be bounded by an arbitrary small constant. Therefore, we impose the condition $T_0 \leq N^{4\sigma-2} T^{-4\varepsilon}$. Note here that, as we assume $\sigma \geq 3/4$, we indeed have $N^{4\sigma-2} T^{-4\varepsilon} \geq Y^{4/3} T^{-4\varepsilon} \geq 1$. Given this condition on T_0 , we get

$$|R| \ll T^\varepsilon N^{2-2\sigma}.$$

We now utilize Huxley's subdivision trick. We divide the interval $[-T, T]$ into $\ll T/T_0$ subintervals of length at most T_0 . Then, the above deduction shows that

$$|R_1| \ll |\tilde{R}_1| \log(Y) \ll T^\varepsilon N^{2-2\sigma} \frac{T}{T_0}.$$

In order to minimize this estimate, we would like to take T_0 as large as possible. Given the above conditions on T_0 , our best option is to choose $T_0 = \min\{2T, N^{4\sigma-2} T^{-4\varepsilon}\}$. Trying both choices and summing the results gives us

$$|R_1| \ll T^\varepsilon (N^{2-2\sigma} + T N^{4-6\sigma}) \ll T^\varepsilon (Y^{4-4\sigma} + T Y^{(16-24\sigma)/3}),$$

since we assume $\sigma \geq 3/4 \geq 2/3$.

This concludes the estimate of $|R_1|$. To bound $|R_2|$, we simply plug $A = 12$ into (5.16) and use Theorem 4.8. We substitute both estimates into (5.8) to get

$$N(\sigma, T) \ll T^\varepsilon (Y^{4-4\sigma} + T Y^{(16-24\sigma)/3} + T^2 Y^{6-12\sigma}).$$

We now aim to fill in $Y = T^{B(\sigma)}$ and optimize $B(\sigma)$. For this purpose, note that the exponents of Y is positive in the first term and non-positive in the other two terms (since $\sigma \geq 3/4$). Hence, we will find the optimal $B(\sigma)$ either by setting the first two terms equal to each other, or the first and the third term, and looking which estimates we get in which intervals of σ . The result of these calculations is that $B(\sigma) = 3/(12\sigma - 4)$ gives the best estimate for all $\sigma \in [2/3, 1]$. Note that for this exponent, $X = T^\varepsilon \leq T^{3/8} \leq Y \leq T^{3/2}$, so the conditions on Y are met. Inserting this exponent gives (5.20). $\square$

Remark 5.5. The above proof shows that the estimate actually holds in the larger interval $2/3 \leq \sigma \leq 1$. However, the estimate of Theorem 5.1 is stronger in $2/3 \leq \sigma \leq 3/4$.

By now, this zero-density estimate has been broken over the whole interval $3/4 \leq \sigma \leq 1$. See [36, Table 11.1] for the current records.

Together with our first estimate in Theorem 5.1, the above bound gives us the constant exponent $A(\sigma) = 12/5$ in (5.1), uniformly in $1/2 \leq \sigma \leq 1$. In particular, we have the following statement.

Theorem 5.6. Fix any $\varepsilon > 0$. Then, uniformly in $1/2 \leq \sigma \leq 1$ and $T \geq 1$, we have

$$N(\sigma, T) \ll T^{\frac{12}{5}(1-\sigma)+\varepsilon} \quad (5.20)$$

Proof. This follows from the previous two zero-density estimates and the fact that

$$\frac{3}{2-\sigma} \leq \frac{12}{5}$$

in $1/2 \leq \sigma \leq 3/4$ and

$$\frac{3}{3\sigma-1} \leq \frac{12}{5}$$

in $3/4 \leq \sigma \leq 1$. □

Note that in the proof, both exponents were maximal in $\sigma = 3/4$. Hence, in order to improve the constant $12/5$, one needs to improve zero-density estimates around this point. Such an improvement was recently established by L. Guth and J. Maynard [13], obtaining the smaller constant $30/13$. It is conjectured that the same estimate holds with the constant 2, which is known as the density hypothesis. In many applications of zero-density estimates, this conjecture provides essentially the same degree of sharpness as the Riemann Hypothesis, despite being a substantially weaker statement. An example of this will be explored in the next chapter.

5.5. Breaking a world record around $\sigma = 9/10$

In this section, we prove a new zero-density estimate that improves the current best bound around the point $\sigma = 9/10$. In particular, we show the following result, which is a new record in the entire stated interval (see [36, Table 11.1] for the previous best results).

Theorem 5.7. Fix any $\varepsilon > 0$. Then we have

$$N(\sigma, T) \ll T^{\frac{64257(1-\sigma)}{81624\sigma-30479}+\varepsilon}, \quad (5.21)$$

uniformly in $0.8916\dots = \frac{823819}{923910} < \sigma < \frac{59727}{66283} = 0.9010\dots$ and $T \geq 1$.

Our method is essentially due to Ivić [22, Thm. 11.2], but we also take inspiration from [9]. Our contribution lies in the optimization of the exponent pair used in (5.17). This is made possible by more recently established exponent pairs summarized in [39].

5.5.1. Ivić's method to bound the contribution of the class-I zeros

Again, the difficulty lies in the estimation of $|R_1|$. We continue with (5.15) and use the Halász-Montgomery inequality in Lemma 5.3. This time we pick

$$\xi(n) = \frac{b(n)}{(e^{-\frac{n}{2N}} - e^{-\frac{n}{N}})^{1/2}}$$

if $N < n \leq 2N$ and zero otherwise, and

$$\varphi_\rho(n) = (e^{\frac{-n}{2N}} - e^{\frac{-n}{N}})^{1/2} n^{-i\gamma}$$

for all positive integers n . Both are elements of the Hilbert space $l^2(\mathbb{N})$. Since the function $f(x) = e^{\frac{-x}{2N}} - e^{\frac{-x}{N}}$ is increasing on $[N, 2N]$, we have $0 < e^{-1/2} - e^{-1} \leq e^{\frac{-n}{2N}} - e^{\frac{-n}{N}} \leq 2$. Therefore,

$$\|\xi\|^2 = \sum_{N < n \leq 2N} \frac{|b(n)|^2}{e^{\frac{-n}{2N}} - e^{\frac{-n}{N}}} \ll N.$$

Hence, the Halász-Montgomery inequality gives us

$$|R|^2 \ll N^{1-2\sigma+2\varepsilon} \sum_{\rho, \rho' \in R} |H(\gamma - \gamma')|, \quad (5.22)$$

where

$$H(t) = \sum_{n=1}^{\infty} (e^{\frac{-n}{2N}} - e^{\frac{-n}{N}}) n^{-it}$$

for any real t . To bound the terms in (5.22) where $\rho = \rho'$, note

$$H(0) = \sum_{n=1}^{\infty} (e^{\frac{-n}{2N}} - e^{\frac{-n}{N}}) \leq \int_0^{\infty} e^{\frac{-x}{2N}} dx \ll N.$$

Therefore, we get

$$|R|^2 \ll N^{1-2\sigma+2\varepsilon} \left(|R|N + \sum_{\rho \neq \rho' \in R} |H(\gamma - \gamma')| \right). \quad (5.23)$$

To estimate the remaining sum, we once again employ the Mellin inversion formula (4.5) for e^{-x} . This yields

$$H(t) = \sum_{n=1}^{\infty} (e^{\frac{-n}{2N}} - e^{\frac{-n}{N}}) n^{-it} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(w+it) \Gamma(w) ((2N)^w - N^w) dw.$$

We change the path of integration to the line $\Re(s) = 1/2$, where we pass one pole at $w = 1 - it$ with residue $O(Ne^{-|t|})$, according to Stirling's formula (1.5). We obtain

$$H(t) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \zeta(w+it) \Gamma(w) ((2N)^w - N^w) dw + O(Ne^{-|t|}).$$

Next, we truncate the integral at $|\Im(w)| = \log^2(T)$. The contribution of the tails is $O(N^{1/2}e^{-\frac{\pi-1}{2}\log^2(T)}) = O(e^{-\log^2(T)}/N)$, again by Stirling's formula (1.5) (also recall here that $N \ll M^k \ll Y^3 \leq T^{3C}$). Summing the result and multiplying by N gives

$$\begin{aligned}
N \sum_{\rho \neq \rho' \in R} |H(\gamma - \gamma')| &\ll N^2 \sum_{\rho \neq \rho' \in R} e^{-|\gamma - \gamma'|} + |R|^2 e^{-\log^2(T)} \\
&\quad + N^{3/2} \sum_{\rho \neq \rho' \in R} \int_{-\log^2 T}^{\log^2 T} |\zeta(1/2 + i\gamma - i\gamma' + iv)| |\Gamma(1/2 + iv)| dv \\
&\ll |R|^2 e^{-\log^2(T)} \\
&\quad + N^{3/2} \sum_{\rho' \in R} \sum_{\rho \in R} \max_{-\log^2(T) \leq v \leq \log^2(T)} |\zeta(1/2 + i\gamma - i\gamma' + iv)|, \quad (5.24)
\end{aligned}$$

where we used the fact that $|\gamma - \gamma'| \geq 5\log^2(T)$ if $\rho \neq \rho' \in R$ to estimate the first term by the second.

Similarly to the proof of Theorem 5.4, we now choose R to be the zeros of $\tilde{R}_1$ with imaginary parts in any subinterval of $[-T, T]$ with length not exceeding $T_0 \in [\log^2(T), 2T]$. Fix $\rho' \in R$ and for every $\rho \in R$, take $v_\rho \in [-\log^2(T), \log^2(T)]$ such that $|\zeta(1/2 + i\gamma - i\gamma' + iv_\rho)|$ is maximal, and denote $\tau_\rho = \gamma - \gamma' + v_\rho$. Note that $|\tau_\rho| \leq 2T_0$ and, by the spacing condition on elements in R_1 , we have $|\tau_{\rho_1} - \tau_{\rho_2}| \geq \log^2(T)$ if $\rho_1 \neq \rho_2$. We now split the above sum according to whether the summands are bigger or smaller than $T_0^{2/13}$, and use Hölder's inequality to obtain

$$\begin{aligned}
&\sum_{\rho \in R} \max_{-\log^2(T) \leq v \leq \log^2(T)} |\zeta(1/2 + i\gamma - i\gamma' + iv)| \\
&\leq \sum_{\substack{\rho \in R \\ |\zeta| \geq T_0^{2/13}}} |\zeta(1/2 + i\tau_\rho)| + \sum_{\substack{\rho \in R \\ |\zeta| < T_0^{2/13}}} |\zeta(1/2 + i\tau_\rho)| \\
&\leq |R|^{5/6} \left(\sum_{\substack{\rho \in R \\ |\zeta| \geq T_0^{2/13}}} |\zeta(1/2 + i\tau_\rho)|^6 \right)^{1/6} + |R|^{18/19} \left(\sum_{\substack{\rho \in R \\ |\zeta| < T_0^{2/13}}} |\zeta(1/2 + i\tau_\rho)|^{19} \right)^{1/19}. \quad (5.25)
\end{aligned}$$

Now we can apply the theory of large values of the Riemann zeta function to obtain a mixed moment estimate. To estimate the first sum, note that since $\mu(1/2) < 1/6$ by Theorem 1.10, we have $|\zeta(1/2 + i\tau_\rho)| < T_0^6$ for all $\rho \in R$ if T is sufficiently large. Applying Corollary 3.9(i) yields

$$\begin{aligned}
\sum_{\substack{\rho \in R \\ |\zeta| \geq T_0^{2/13}}} |\zeta(1/2 + i\tau_\rho)|^6 &\leq \sum_{1 \leq 2^l \leq T_0^{1/78}} \sum_{2^l T_0^{2/13} \leq |\zeta| < 2^{l+1} T_0^{2/13}} |\zeta(1/2 + i\tau_\rho)|^6 \\
&\ll \sum_{1 \leq 2^l \leq T_0^{1/78}} \left(\frac{T_0}{2^{6l} T_0^{12/13}} + \frac{T_0^3}{2^{19l} T_0^{38/13}} \right) \log^{49/2}(T_0) (2^{l+1} T_0^{2/13})^6
\end{aligned}$$

$$\begin{aligned}
&\ll T_0 \log^{49/2}(T_0) \sum_{1 \leq 2^l \leq T_0^{1/78}} 1 \\
&\ll T_0 \log^{51/2}(T).
\end{aligned}$$

For the second term, we take a similar approach, where we use the inequality $|R| \leq T_0$ that follows because the zeros in R are well-spaced, to see that

$$\begin{aligned}
\sum_{\substack{\rho \in R \\ |\zeta| < T_0^{2/13}}} |\zeta(1/2 + i\tau_\rho)|^{19} &\leq \sum_{\substack{\rho \in R \\ |\zeta| \leq 1}} |\zeta(1/2 + i\tau_\rho)|^{19} + \sum_{1 \leq 2^l \leq T_0^{2/13}} \sum_{\substack{\rho \in R \\ \frac{T_0^{2/13}}{2^{l+1}} \leq |\zeta| < \frac{T_0^{2/13}}{2^l}}} |\zeta(1/2 + i\tau_\rho)|^{19} \\
&\leq |R| + \sum_{1 \leq 2^l \leq T_0^{2/13}} \left(\frac{2^{6l+6} T_0}{T_0^{12/13}} + \frac{2^{19l+19} T_0^3}{T_0^{38/13}} \right) \log^{49/2}(T_0) \left(\frac{T_0^{2/13}}{2^l} \right)^{19} \\
&\leq T_0 + T_0^3 \log^{49/2}(T_0) \sum_{1 \leq 2^l \leq T_0^{2/13}} 1 \\
&\ll T_0^3 \log^{51/2}(T).
\end{aligned}$$

We conclude from (5.24) and the above discussion that for our choice of R , we have

$$N \sum_{\rho \neq \rho' \in R} |H(\gamma - \gamma')| \ll |R|^2 e^{-\log^2(T)} + N^{3/2} |R| T^\varepsilon (|R|^{5/6} T_0^{1/6} + |R|^{18/19} T_0^{3/19}).$$

Plugging this into (5.23) gives

$$|R|^2 \ll T^\varepsilon \left(|R| N^{2-2\sigma} + |R|^2 e^{-\log^2(T)} + N^{\frac{3}{2}-2\sigma} |R|^{11/6} T_0^{1/6} + N^{\frac{3}{2}-2\sigma} |R|^{37/19} T_0^{3/19} \right).$$

Once T is large enough, we can move the second term to the left-hand side and obtain

$$|R|^2 \ll T^\varepsilon \left(|R| N^{2-2\sigma} + N^{\frac{3}{2}-2\sigma} |R|^{11/6} T_0^{1/6} + N^{\frac{3}{2}-2\sigma} |R|^{37/19} T_0^{3/19} \right). \quad (5.26)$$

We now consider three cases, one for each term on the right-hand side.

- If the first term is the largest one, then $|R|^2 \ll T^\varepsilon |R| N^{2-2\sigma}$, so $|R| \ll T^\varepsilon N^{2-2\sigma}$.
- If the second term is the largest one, then $|R|^2 \ll T^\varepsilon N^{\frac{3}{2}-2\sigma} |R|^{11/6} T_0^{1/6}$, so $|R| \ll T^\varepsilon N^{9-12\sigma} T_0$.
- If the third term is the largest one, then $|R|^2 \ll T^\varepsilon N^{\frac{3}{2}-2\sigma} |R|^{37/19} T_0^{3/19}$, so $|R| \ll T^\varepsilon N^{\frac{57}{2}-38\sigma} T_0^3$.

In any case, we have

$$|R| \ll T^\varepsilon \left(N^{2-2\sigma} + N^{9-12\sigma} T_0 + N^{\frac{57}{2}-38\sigma} T_0^3 \right).$$

The parameter N here still depends on which interval we choose of R (and therefore also on T_0). However, we can resolve this similarly as in the proof of Theorem 5.4. Recall from subsection 5.3.1 that $M^k \leq N(R) \leq 2^k M^k$ and $k \ll 1$, so $N(\tilde{R}_1) \ll N(R) \ll N(\tilde{R}_1)$. In the remainder of this section, N will always denote $N(\tilde{R}_1)$, and the previous sentence shows that (5.26) still holds this way.

We finally apply Huxley's subdivision trick. As before, we divide the interval $[-T, T]$ into $\ll T/T_0$ subintervals of length at most T_0 . Summing the contributions over all subintervals gives

$$\begin{aligned} |R_1| &\ll |\tilde{R}_1| \log(Y) \ll \frac{T}{T_0} T^\varepsilon \left(N^{2-2\sigma} + N^{9-12\sigma} T_0 + N^{\frac{57}{2}-38\sigma} T_0^3 \right) \\ &= T^\varepsilon \left(N^{2-2\sigma} T T_0^{-1} + N^{9-12\sigma} T + N^{\frac{57}{2}-38\sigma} T T_0^2 \right). \end{aligned}$$

Only two terms involve T_0 , and the corresponding exponents of T_0 have opposite signs. Therefore, in order to optimize our choice of T_0 , we set both terms equal and calculate the resulting value of T_0 , which gives $T_0 = N^{\frac{72\sigma-53}{6}}$. However, we imposed the condition that $\log^2(T) \leq T_0 \leq 2T$. The first inequality can easily be seen to hold once $\sigma \geq 3/4 > 53/72$. We can afford this decision since we are trying to find a zero-density estimate around $\sigma = 9/10$. The second inequality cannot easily be guaranteed, which is why we choose $T_0 = \min\{T, N^{\frac{72\sigma-53}{6}}\}$. If the minimum is T , we have under the new condition $\sigma \leq 11/12$ that

$$\begin{aligned} |R_1| &\ll T^\varepsilon \left(N^{2-2\sigma} + N^{9-12\sigma} T + N^{\frac{57}{2}-38\sigma} T^3 \right) \\ &\leq T^\varepsilon \left(N^{2-2\sigma} + N^{\frac{65-84\sigma}{6}} T + N^{\frac{57}{2}-38\sigma} \left(N^{\frac{72\sigma-53}{6}} \right)^2 T \right) \\ &\ll T^\varepsilon \left(N^{2-2\sigma} + N^{\frac{65-84\sigma}{6}} T \right). \end{aligned}$$

On the other hand, if the minimum is $N^{\frac{72\sigma-53}{6}}$, we similarly have (again under the condition $\sigma \leq 11/12$) that

$$|R_1| \ll T^\varepsilon N^{\frac{65-84\sigma}{6}} T \leq T^\varepsilon \left(N^{2-2\sigma} + N^{\frac{65-84\sigma}{6}} T \right).$$

We conclude that the same estimate is reached in both cases.

In order to make the exponent in the second term on the right-hand side non-positive (which is the case around $\sigma = 9/10$), we assume the slightly stronger condition $\sigma \geq 65/84$. Since $Y^{4/3} \leq M^k \leq N \leq 2^k M^k \ll Y^2 \log^4(Y)$, we conclude with

$$|R_1| \ll T^\varepsilon \left(Y^{4-4\sigma} + Y^{\frac{130-168\sigma}{9}} T \right). \quad (5.27)$$

5.5.2. Optimizing the exponent pair

For our final zero-density estimate, we combine (5.27) with (5.17) into (5.8), where the exponent pair (κ, λ) remains to be chosen. This yields

$$N(\sigma, T) \ll T^\varepsilon (Y^{4-4\sigma} + TY^{\frac{130-168\sigma}{9}} + TY^{3-6\sigma} + T^{\frac{\kappa+\lambda}{\kappa}} Y^{(1-2\sigma)\frac{1+2\kappa+2\lambda}{\kappa}}) \quad (5.28)$$

in $65/84 \leq \sigma \leq 11/12$. In [22, Thm. 11.2], Ivić chose the exponent pair $(2/7, 4/7)$, giving

$$N(\sigma, T) \ll T^\varepsilon (Y^{4-4\sigma} + TY^{\frac{130-168\sigma}{9}} + TY^{3-6\sigma} + T^3 Y^{\frac{19}{2}(1-2\sigma)}). \quad (5.29)$$

As before, we set $Y = T^{B(\sigma)}$ and optimize $B(\sigma)$ by setting the first term (with non-negative exponent for Y) equal to one of the other terms (with non-positive exponent for Y). It turns out that, in an interval around $\sigma = 9/10$, the estimate becomes optimal by comparing the first term with the last. This results in the choice $Y = T^{\frac{6}{30\sigma-11}}$, which gives the estimate

$$N(\sigma, T) \ll T^{\frac{24(1-\sigma)}{30\sigma-11} + \varepsilon} \quad (5.30)$$

in $\frac{155}{174} \leq \sigma \leq \frac{11}{12}$. This new lower bound on σ ensures that the second term in (5.29) is smaller than the right-hand side of (5.30).

Since Ivić established this result, new exponent pairs have been found. Our job is now to find the optimal currently proven exponent pair (κ, λ) to make the estimate (5.28) as small as possible. We choose among the recent exponent pairs described in Section 3.2.

Again, we substitute $Y = T^{B(\sigma)}$ into (5.28) and equate the first and last terms. This gives the proposed exponent

$$B(\sigma) = \frac{\kappa + \lambda}{2\kappa - 1 - 2\lambda + (2 + 4\lambda)\sigma}. \quad (5.31)$$

The exponent of T in both terms then becomes

$$\frac{4(\kappa + \lambda)(1 - \sigma)}{2\kappa - 1 - 2\lambda + (2 + 4\lambda)\sigma}. \quad (5.32)$$

In order to make our calculations more manageable, we fix the value $\sigma = 9/10$ (which is contained in the interval of possible improvement). This gives the function

$$f(\kappa, \lambda) = \frac{\kappa + \lambda}{2 + 5\kappa + 4\lambda}$$

that we need to minimize by choosing an optimal exponent pair (κ, λ) .

To gain further insight, we plot the lines corresponding to $f(\kappa, \lambda) = c$ for various values of c over Figure 3.1, which displays the exponent pairs under consideration. The result is shown in Figure 5.1 for 81 evenly spaced values $c \in [0.12, 0.17]$. As we move further down or to the left, the values of c decrease.

A visual argument clearly shows that the smallest value for c is reached at the boundary of the convex hull of these exponent pairs, and thus in one of the vertices. From the figure, we expect the minimum to be reached in (x_{-2}, y_{-2}) , (x_{-3}, y_{-3}) , or (x_{-4}, y_{-4}) .

To determine the best exponent pair, we use the Python code of Listing A.1 in the appendix. This algorithm generates the exponent pairs (x_n, y_n) for $|n| \leq 20$, evaluates $f(\kappa, \lambda)$ in each of these pairs, and selects all minimizers. The only result is the exponent pair

$$(x_{-3}, y_{-3}) = \left(\frac{10333}{38033} + \varepsilon, \frac{43591}{76066} + \varepsilon \right), \quad (5.33)$$

which we will use in the remainder of this section.

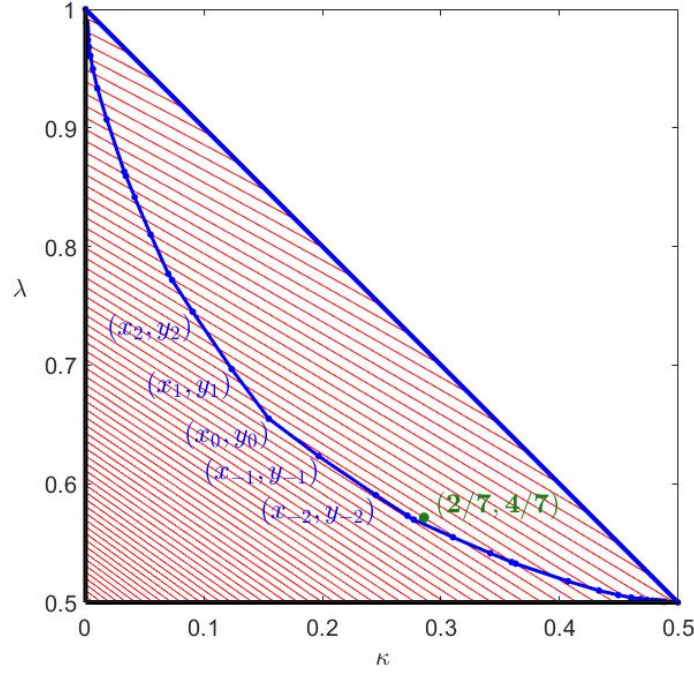


Figure 5.1: Level curves of $f(\kappa, \lambda)$ (in red), together with the exponent pairs of Figure 3.1 (in blue). The previously used exponent pair $(2/7, 4/7)$ is highlighted in green.

5.5.3. Final calculations

We use $Y = T^{B(\sigma)}$ in (5.29) for the exponent pair (x_{-3}, y_{-3}) , with $B(\sigma)$ as in (5.31). Note here that $T^{B(1)} \leq Y \leq T^{B(1/2)}$ and $B(1/2), B(1) > 0$, so the condition $X = T^\varepsilon \leq Y \leq T^C$ is satisfied for $C = B(1/2)$. We can calculate the exponent of the first and last terms with (5.32). This yields

$$N(\sigma, T) \ll T^\varepsilon \left(T^{\frac{64257(1-\sigma)}{81624\sigma-30479}} + T^{\frac{1209361-1309452\sigma}{6 \cdot (81624\sigma-30479)}} + T^{\frac{70855-59046\sigma}{4 \cdot (81624\sigma-30479)}} \right). \quad (5.34)$$

Out of these three terms, the first one will be the largest in the interval $0.8916\dots = \frac{823819}{923910} \leq \sigma \leq \frac{186173}{197982} = 0.9403\dots$, which contains $\sigma = 9/10$. This means that, in this interval, substituting the value for Y that we get from equating the first and last terms in (5.28) gives an optimal estimate. In conclusion, we have

$$N(\sigma, T) \ll T^{\frac{64257(1-\sigma)}{81624\sigma-30479} + \varepsilon} \quad (5.35)$$

in the interval $0.8916\dots = \frac{823819}{923910} \leq \sigma \leq \frac{11}{12} = 0.9166\dots$

We end this section by looking in which interval of σ this improves the current best zero-density estimates. Around $\sigma = 9/10$, these records (see e.g. [36, Table 11.1]) include (5.30) in $0.8885\dots = 279/314 \leq \sigma \leq 9/10$ [22, 9] and

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{10\sigma-7} + \varepsilon} \quad (5.36)$$

in $9/10 \leq \sigma \leq 31/34 = 0.9117\dots$ [35, Thm. 50]. By comparing exponents, we find that (5.35) refines these estimates in $0.8916\dots = \frac{823819}{923910} < \sigma < \frac{59727}{66283} = 0.9010\dots$, leading to the formulation of Theorem 5.7.

Chapter 6

Applications to primes in short intervals

As an application of zero-density estimates, we now present a short chapter on the theory of primes in short intervals. In this setting, a short interval refers to an interval of the form $(x, x + y]$, where $0 \leq y \leq x$. The goal is to obtain a good estimate for the number of primes in such an interval. This number can be quantified with the prime counting function as $\pi(x + y) - \pi(x)$.

However, it is usually easier to work with the closely related Chebyshev function, defined as

$$\psi(x) = \sum_{p^k \leq x} \log(p),$$

where the sum runs over all prime powers not exceeding x (with $k \geq 1$). The connection between both functions becomes apparent through partial summation, as illustrated in the proof of Corollary 6.3 below. In particular, the Chebyshev function has an explicit formula, of which the approximate variant is given by the following theorem. This will be the starting point for the proof of the main theorem in this chapter. Its proof relies on Perron's formula.

Theorem 6.1 (see e.g. [37, Thm. 4.6]) Uniformly for $x \geq 2$ and $2 \leq T \leq x$, we have

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T} \log^2(x)\right),$$

where the sum runs over non-trivial zeros of $\zeta(s)$, counted with multiplicity.

One therefore aims to estimate the function

$$\psi(x + y) - \psi(x).$$

We first discuss the estimate that the prime number theorem provides, after which zero-density estimates will be applied to gain a significant improvement. We end with a corollary that bounds gaps between consecutive primes.

6.1. An estimate via the prime number theorem

One of the many equivalent formulations of the prime number theorem is an estimate of the form

$$\psi(x) = x + O(E(x)), \quad (6.1)$$

where $E(x)$ is an increasing function. A central problem in analytic number theory is to improve this error term. If we plug in $x + y$ and x into this formula and subtract both results, we obtain

$$\psi(x + y) - \psi(x) = y + O(E(2x)). \quad (6.2)$$

This estimate is mostly useful in the range $E(2x) \leq y \leq x$, as otherwise the error term dominates.

Currently, the best known error term in (6.1) is given by

$$E(x) = x \exp \left(-A \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right) \quad (6.3)$$

for some $A > 0$. This can be proven via the explicit formula of $\psi(x)$ and the current best zero-free region of the zeta function given in Theorem 1.5 (see e.g. [22, Thm. 12.2]). Another method, based on Perron's formula, can be found in [37, Thm. 4.1]. In both methods, a stronger zero-free region immediately implies a stronger error term.

Substituting the function $E(x)$ into (6.2) yields

$$\psi(x + y) - \psi(x) = y + O \left(x \exp \left(-A \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right) \right), \quad (6.4)$$

where a smaller value for A was chosen. Since the error term is $\gg_\varepsilon x^{1-\varepsilon}$, this statement is relatively weak compared to the main theorem in the next section, which does not rely on the prime number theorem.

We end this section by remarking that the Riemann Hypothesis does give a good estimate. In particular, it implies that the error term $E(x) = x^{1/2} \log^2(x)$ holds in (6.1), and therefore also in (6.4). This can be achieved using Theorem 1.6 combined with a partial summation argument.

6.2. A stronger result using zero-density estimates

We now turn to the main theorem of this chapter, which establishes a much better estimate than (6.4). We use the Hoheisel–Ingham method, as described in [23], which exploits two main results of the zeta function. First, we will consider any fixed $c \geq 2$ such that

$$N(\sigma, T) \ll_\varepsilon T^{c(1-\sigma)+\varepsilon} \quad (6.5)$$

holds uniformly in $1/2 \leq \sigma \leq 1$. An estimate of this form was achieved in Theorem 5.6 for $c = 12/5$. Note that within any range of the form $1/2 \leq \sigma \leq \sigma_0 < 1$, this is equivalent to the estimate

$$N(\sigma, T) \ll_{\sigma_0, \varepsilon} T^{(c+\varepsilon)(1-\sigma)}.$$

In the range $\sigma_0 \leq \sigma \leq 1$, we use the different bound

$$N(\sigma, T) \ll_{\sigma_0, B} T^{(1-\sigma)} \log^B(T),$$

where $B \geq 0$. Such an estimate holds, for example, for $\sigma_0 = 152/155$ and $B = 16$, see [22, Thm. 11.3]. Together, these two results yield

$$N(\sigma, T) \ll_{\varepsilon} T^{(c+\varepsilon)(1-\sigma)} \log^B(T), \quad (6.6)$$

where we fixed the parameters B and σ_0 .

The second main result that we use is the zero-free region in Theorem 1.5. However, we do not need the current best zero-free region to obtain a good estimate. In fact, one can follow the same proof with Littlewood's zero-free region [26] in order to obtain the same result, but with a slightly worse error term and a different value of θ .

Theorem 6.2. Fix $c \geq 2$ and $B \geq 0$ such that (6.6) holds, $A > 0$ as in Theorem 1.5, and $\varepsilon > 0$. Define $\theta = 1 - (c + 2\varepsilon/A)^{-1}$ and $\delta = \varepsilon(1 - \theta)^{1/3}$. Then

$$\psi(x+y) - \psi(x) = y + O\left(y \exp\left[-\delta \left(\frac{\log x}{\log \log x}\right)^{1/3}\right]\right) \quad (6.7)$$

holds uniformly for $x \geq 3$ and $x^\theta \leq y \leq x$.

Proof. Since $\psi(x)$ is a step function and the remaining terms in (6.7) are continuous, we may assume that x is sufficiently large. Next, we apply Theorem 6.1 to $x+y$ and x , choosing $T = x^{1-\theta+\varepsilon'}$ in both cases. Here, we have fixed some $\varepsilon' > 0$, which we assume to be sufficiently small throughout this proof. Subtracting both results and dividing by y yields

$$\frac{\psi(x+y) - \psi(x)}{y} - 1 = - \sum_{|\gamma| \leq T} \frac{(x+y)^\rho - x^\rho}{\rho y} + O\left(x^{-\varepsilon'/2}\right).$$

This formulation allows us to use Lagrange's mean value theorem for all terms in the sum, but one needs to consider the real and imaginary parts separately. We find $a, b \in (x, x+y)$ such that

$$\frac{(x+y)^\rho - x^\rho}{y} = \Re(\rho a^{\rho-1}) + i \Im(\rho b^{\rho-1}) \ll (|\rho a^{\rho-1}|^2 + |\rho b^{\rho-1}|^2)^{1/2} \ll |\rho| x^{\beta-1}.$$

Therefore,

$$\frac{\psi(x+y) - \psi(x)}{y} - 1 \ll x^{-\varepsilon'/2} + \sum_{|\gamma| \leq T} |\rho| x^{\beta-1},$$

so it suffices to adequately estimate the sum.

In order to incorporate zero-density estimates, we introduce a Riemann–Stieltjes integral and use integration by parts to obtain

$$\sum_{|\gamma| \leq T} x^{\beta-1} = \int_0^1 x^{\alpha-1} d(-N(\alpha, T))$$

$$\begin{aligned}
&= x^{-1}N(0, T) + \log(x) \int_0^1 x^{\alpha-1} N(\alpha, T) d\alpha \\
&\leq x^{-1}N(0, T) + \frac{1}{2} \log(x) x^{-1/2} N(0, T) + \log(x) \int_{1/2}^1 x^{\alpha-1} N(\alpha, T) d\alpha, \quad (6.8)
\end{aligned}$$

where we split the integral into two intervals and trivially estimated the first part. An application of the Riemann–von Mangoldt formula (1.7) yields a sufficient estimate for the first two terms. Namely, since $\theta > 1/2$, we have

$$x^{-1}N(0, T) + \frac{1}{2} \log(x) x^{-1/2} N(0, T) \ll \log(x) x^{-1/2} T \log(T) \ll x^{1/2-\theta} \log^2(x) \ll x^{-\varepsilon'}.$$

Denote $\eta = A/((\log T)^{2/3}(\log \log T)^{1/3})$. Using the zero-free region in Theorem 1.5, we have $N(\alpha, T) = 0$ if $\alpha \geq 1 - \eta$. Therefore, we ignore this part of the integral in (6.8), and substitute the zero-density estimate (6.6). This leads to

$$\begin{aligned}
\log(x) \int_{1/2}^1 x^{\alpha-1} N(\alpha, T) d\alpha &\ll \log(x) \int_{1/2}^{1-\eta} x^{\alpha-1} T^{(c+\varepsilon')(1-\alpha)} \log^B(T) d\alpha \\
&\ll \log^B(T) \log(x) \int_{\eta}^{1/2} (T^{c+\varepsilon'}/x)^{\alpha} d\alpha \\
&\ll \log^B(T) (T^{c+\varepsilon'}/x)^{\eta}.
\end{aligned}$$

It remains to substitute the values of η and θ and compute

$$(T^{c+\varepsilon'}/x)^{\eta} = T^{(c+\varepsilon' - \frac{1}{1-\theta}) \frac{A}{(\log T)^{2/3}(\log \log T)^{1/3}}} \leq e^{-\varepsilon \left(\frac{\log T}{\log \log T} \right)^{1/3}}.$$

Finally, by substituting $T = x^{1-\theta+\varepsilon'}$, we find

$$\log^B(T) e^{-\varepsilon \left(\frac{\log T}{\log \log T} \right)^{1/3}} \ll e^{-\delta \left(\frac{\log x}{\log \log x} \right)^{1/3}}. \quad \square$$

Note that in this theorem, the values of A and B are irrelevant, since we may choose an arbitrary $\varepsilon > 0$. In fact, we can conclude that for any $\theta > 1 - 1/c$, there exists some $\delta > 0$ such that (6.7) holds in the region defined by $x \geq 3$ and $x^{\theta} \leq y \leq x$. The value for c obtained in Theorem 5.6 allows us to choose any $\theta > 7/12$. However, due to the recent improvement of the constant c by Guth and Maynard [13], this can be lowered to $\theta > 17/30$. This is, as far as the author's knowledge goes, the best result to date.

The density hypothesis, which claims that the estimate (6.5) holds for the constant $c = 2$, implies that (6.7) holds in the region $x \geq 3$ and $x^{1/2+\varepsilon} \leq y \leq x$ (where the constant of the O -term may depend on ε). This domain is almost as good as the one implied by the Riemann Hypothesis in the previous section. However, the error term implied by this conjecture is independent of y , and therefore cannot be matched by the density hypothesis. In order to break the barrier at $\theta = 1/2$, if possible, a more advanced argument will be needed.

We end this section by transferring the above result to the prime counting function, but at the cost of a slightly weaker error term.

Corollary 6.3. For any fixed $\varepsilon > 0$, we have

$$\pi(x+y) - \pi(x) = \frac{y}{\log(x)} + O\left(\frac{y}{\log^2(x)}\right)$$

holds uniformly for $x \geq 3$ and $x^{17/30+\varepsilon} \leq y \leq x$.

Proof. We begin by defining the function

$$\vartheta(x) = \sum_{p \leq x} \log(p),$$

which is related to the Chebyshev function by the estimate

$$\psi(x) - \vartheta(x) = \sum_{\substack{p^k \leq x \\ k \geq 2}} \log(p) \leq \sum_{n \leq \sqrt{x}} \log(x) \leq \sqrt{x} \log(x).$$

In particular, the above discussion implies that there exists a $\delta > 0$ such that

$$\vartheta(x+y) - \vartheta(x) = \psi(x+y) - \psi(x) + O(\sqrt{x} \log(x)) = y + O\left(y e^{-\delta\left(\frac{\log x}{\log \log x}\right)^{1/3}}\right) \quad (6.9)$$

holds uniformly for $x \geq 3$ and $x^{17/30+\varepsilon} \leq y \leq x$. Moreover, the prime number theorem in the form (6.1) with the error term (6.3) implies $\vartheta(x) = O(x)$.

The next step is to apply partial summation, giving

$$\begin{aligned} \pi(x+y) - \pi(x) &= \sum_{x < p \leq x+y} \log(x) \cdot \frac{1}{\log(x)} \\ &= \frac{\vartheta(x+y)}{\log(x+y)} - \frac{\vartheta(x)}{\log(x)} + \int_x^{x+y} \frac{\vartheta(t)}{t \log^2(t)} dt \\ &= \frac{\vartheta(x+y) - \vartheta(x)}{\log(x)} + \vartheta(x+y) \left(\frac{1}{\log(x+y)} - \frac{1}{\log(x)} \right) + \int_x^{x+y} \frac{\vartheta(t)}{t \log^2(t)} dt. \end{aligned}$$

For the first term, we substitute (6.9), after which it remains to estimate the other two terms by $y/\log^2(x)$. In the second term, we apply Lagrange's mean value theorem to find some $c \in (x, x+y)$ such that

$$\vartheta(x+y) \left(\frac{1}{\log(x+y)} - \frac{1}{\log(x)} \right) = -\vartheta(x+y) \frac{y}{c \log^2(c)} = O\left(\frac{y}{\log^2(x)}\right).$$

Lastly, the third term can be estimated by

$$\int_x^{x+y} \frac{1}{\log^2(t)} dt \leq \frac{y}{\log^2(x)}.$$

□

6.3. A consequence for gaps between primes

Denote the n -th prime by p_n . An easy consequence of Corollary 6.3 is an estimate for the gaps between consecutive primes.

Theorem 6.4. For any fixed $\varepsilon > 0$, we have

$$p_{n+1} - p_n \ll p_n^{17/30+\varepsilon} \ll n^{17/30+2\varepsilon}.$$

Proof. Denote $\theta = 17/30 + \varepsilon$. We apply Corollary 6.3 to $x = p_n$ and $y = x^\theta$. This yields

$$\pi(p_n + p_n^\theta) - \pi(p_n) \gg \frac{p_n^\theta}{\log(p_n)} \gg 1, \quad n \rightarrow \infty.$$

Hence, if n is sufficiently large, there must be a prime in the interval $(p_n, p_n + p_n^\theta]$. In particular, it contains the $(n+1)$ -th prime, so

$$p_{n+1} - p_n \ll p_n^\theta.$$

To obtain the second estimate, recall that by the prime number theorem, $\pi(x) \sim x/\log(x)$. This can, for example, be obtained from (6.1) with the error term (6.3) by partial summation. In particular, substituting $x = p_n$ gives $n \gg p_n/\log(p_n)$, so $n^{1+\varepsilon'} \gg p_n$ for any fixed $\varepsilon' > 0$. Therefore,

$$p_n^{17/30+\varepsilon} \ll n^{(1+\varepsilon')(17/30+\varepsilon)} \ll n^{17/30+2\varepsilon}$$

if ε' is small enough. □

This is, however, not the best proven estimate. The current record is held by Baker, Harman and Pintz [1], who proved

$$p_{n+1} - p_n \ll p_n^{0.525}.$$

The density hypothesis implies the bound $p_{n+1} - p_n \ll p_n^{1/2+\varepsilon}$, with the same proof as before, while the Riemann Hypothesis implies the bound $p_{n+1} - p_n \ll p_n^{1/2} \log(p_n)$. The latter estimate needs a slightly more adapted argument. The Riemann Hypothesis implies the error term $E(x) = x^{1/2} \log^2(x)$ in (6.1), and therefore

$$\pi(x) = \int_2^x \frac{1}{\log(t)} dt + O(x^{1/2} \log(x))$$

with a partial summation argument. Then, analogously as how we obtained (6.4), we have

$$\pi(x+y) - \pi(x) = \int_x^{x+y} \frac{1}{\log(t)} dt + O(x^{1/2} \log(x))$$

for $x \geq 3$ and $x^{1/2} \log(x) \leq y \leq x$.

With a probabilistic model for primes, Cramér [11] conjectured the much stronger bound $p_{n+1} - p_n \ll \log^2(p_n)$ in 1937. As a lower bound, Ford, Green, Konyagin, Maynard and Tao showed in 2018 [12] that

$$p_{n+1} - p_n \gg \frac{(\log p_n)(\log \log p_n)(\log \log \log \log p_n)}{\log \log \log p_n}$$

for infinitely many n . This result earned them Erdős' prize of \$10,000, the largest he ever offered for a solution to a mathematical problem.

Conclusion

After introducing the necessary tools from analytic number theory, part II of this thesis can be summarized into three main topics. First, we studied moment estimates for the Riemann zeta function on the critical line, proving an asymptotic formula for the second and fourth moment. Moreover, we proved essentially optimal estimates for exponents $0 \leq A \leq 4$, and the current best bounds in $4 < A \leq 12$. Secondly, we introduced the zero detection method used in zero-density estimates and applied it to prove the current best bounds in the interval $1/2 \leq \sigma \leq 7/10$. In addition, we succeeded in our goal to improve current zero-density estimates, breaking a world record in the interval $0.8916\dots < \sigma < 0.9010\dots$. Lastly, we illustrated the importance of this work by proving an estimate for the number of primes in a short interval, which provided a much better result than the best known error term in the prime number theorem.

It should be noted that the method in Section 5.5 offers more flexibility than was exploited in this thesis, presenting the opportunity to break more world records. First of all, the choice of $\sigma = 9/10$ is quite arbitrary, and one could easily try substituting different values and applying the same strategy. However, as we saw with the value $\sigma = 9/10$, an optimal estimate will be reached in one of “the current best known exponent pairs”, meaning one of the vertices in the infinite polygon of Figure 3.1. Therefore, it would be more useful to consider which of these exponent pairs provide an optimal estimate in which intervals for σ . Another option is to also consider the intervals where the other terms of (5.28) take over. For example, in (5.34), the second term will be the largest one if $\sigma < 0.8916\dots$ and, once we ignore the other terms, the resulting estimate still improves upon Ivić’s estimate (5.30) in a small interval of σ . Finally, the mixed moment estimate that was used to bound (5.25) relied on the exponent pair $(2/7, 4/7)$ via Theorem 3.8, and one may continue the argument without picking a specific exponent pair in order to optimize it later as well.

We would also like to stress that the exponent pairs that we considered in Section 3.2 are not the most recent anymore. In 2025, new exponent pairs have been found, see [35, Thm. 20] and [36, Thm. 5.23]. Therefore, we may apply the same ideas with these newer exponent pairs to possibly refine our result further. Furthermore, to extend the interval of validity in Theorem 5.7, one can incorporate Bourgain’s dichotomy technique [6], as in [9].

Appendix

Listing A.1: Python code used to determine the optimal exponent pair in Subsection 5.5.2.

```
1 from fractions import Fraction
2
3
4
5 # The function f(kappa,lambda), i.e. the exponent to be minimized in
   sigma = 9/10.
6 def f(k,l) :
7     return (k+1)/((2+5*k+4*l))
8
9 # The function A(kappa,lambda).
10 def A(k,l) :
11     return [k/((2*k + 2) , 1/((2*k + 2) + Fraction(1,2))]
12
13 # The function B(kappa,lambda).
14 def B(k,l) :
15     return [1 - Fraction(1,2) , k + Fraction(1,2)]
16
17 # This function generates the exponent pairs (x_n,y_n) for |n| <= m. For
   every exponent pair, we save its x-coordinate, its y-coordinate and
   its index n.
18 def generate_exp_pairs(m) :
19
20     # The first few exponent pairs for n = 0,1,2,3,4.
21     generates = [[Fraction(13, 84), Fraction(55, 84), 0] , [Fraction
   (4742, 38463), Fraction(35731, 51284), 1] , [Fraction(18, 199),
   Fraction(593, 796), 2] , [Fraction(2779, 38033), Fraction(58699,
   76066), 3] , [Fraction(715, 10238), Fraction(7955, 10238), 4]]
22
23     # The exponent pairs for n >= 5.
24     for j in range(5 , m+1):
25         if j <= 8 : # The exponent pairs for n = 5,6,7,8.
26             new_pair = A(generates[j-4][0] , generates[j-4][1])
27             new_pair.append(j)
28         else : # The exponent pairs for n >= 9.
29             new_pair = [Fraction( 2 , ((j-5)**2) * (j-2) ) , 1 -
   Fraction( 3*(j-4) - 2 , (j-5)*(j-4)*(j-2) ) , j]
30             generates.append(new_pair)
31
32     # The exponent pairs for n <= -1.
33     for j in range(1,m+1) :
34         new_pair = B(generates[2*j-1][0] , generates[2*j-1][1])
35         new_pair.append(-j)
```

```

36     generates.insert(0,new_pair)
37
38     return generates
39
40
41
42 # We generate the exponent pairs for |n| <= 20 and save the ones for
43   which f(kappa,lambda) is minimal.
44 the_pairs = generate_exp_pairs(20)
45
46 current_best = []
47
48 for pair in the_pairs :
49     if len(current_best) == 0 :
50         current_best.append(pair)
51     elif f(pair[0] , pair[1]) < f(current_best[0][0] , current_best
52       [0][1]) :
53         current_best = [pair]
54     elif f(pair[0] , pair[1]) == f(current_best[0][0] , current_best
55       [0][1]) :
56         current_best.append(pair)
57
58
59 # We print some data about all optimal choices.
60 print("The following exponent pairs are optimal:")
61 print(" ")
62 for choice in current_best :
63     k = choice[0]
64     l = choice[1]
65     index = choice[2]
66     print("Index:", index)
67     print("Exponent pair:", [k,l])
68     print("Numerically:", [float(k), float(l)])
69     print(" ")
70
71 print(" ")
72 print(" ")
73 print("In these points, the minimal value of f(kappa,lambda) becomes:",
74       f(current_best[0][0], current_best[0][1]))
75 print("Numerically:", float(f(current_best[0][0], current_best[0][1])))

```

References

- [1] R. C. Baker, G. Harman and J. Pintz, *The difference between consecutive primes*, Proceedings of the London Mathematical Society, vol. 83, no. 3, pp. 532–562, 2001.
- [2] D. Bhatta and L. Debnath, *Integral transforms and their applications*, 3rd ed., CRC Press, 2015.
- [3] Y. Bicheng and C. Qiang, *A survey on the study of Hilbert-type inequalities*, Journal of Inequalities and Applications, vol. 2015, Article no. 302.
- [4] E. Bombieri and H. Iwaniec, *On the order of $\zeta(1/2 + it)$* , Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Serie 4, vol. 13, no. 3, pp. 449–472, 1986.
- [5] E. Bombieri and H. Iwaniec, *Some mean-value theorems for exponential sums*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Serie 4, vol. 13, no. 3, pp. 473–486, 1986.
- [6] J. Bourgain, *On large values estimates for Dirichlet polynomials and the density hypothesis for the Riemann Zeta function*, International Mathematics Research Notices, vol. 3, pp. 133–146, 2000.
- [7] J. Bourgain, *Decoupling, exponential sums and the Riemann zeta function*, Journal of the American Mathematical Society, vol. 30, no. 1, pp. 205–224, 2017.
- [8] E. Carneiro and F. Littmann, *Monotone extremal functions and the weighted Hilbert’s inequality*, Portugaliae Mathematica, vol. 81, no. 1/2, pp. 127–134, 2024.
- [9] B. Chen, G. Debruyne, and J. Vindas, *On the density hypothesis for L -functions associated with holomorphic cusp forms*, Revista Matemática Iberoamericana, vol. 40, no. 6, pp. 2179–2202, 2024.
- [10] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Integral moments of L -functions*, Proceedings of the London Mathematical Society, vol. 91, no. 1, pp. 33–104, 2005.
- [11] H. Cramér, *On the order of magnitude of the difference between consecutive prime numbers*, Prace Matematyczno-Fizyczne, vol. 45, no. 1, pp. 51–74, 1937.
- [12] K. Ford, B. Green, S. Konyagin, J. Maynard and T. Tao, *Long gaps between primes*, Journal of the American Mathematical Society, vol. 31, no. 1, pp. 65–105, 2018.
- [13] L. Guth and J. Maynard, *New large value estimates for Dirichlet polynomials*, Annals of Mathematics, vol. 203, no. 2, pp. 623–675, 2026.

- [14] G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Mathematica, vol. 41, pp. 119–196, 1918.
- [15] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1934.
- [16] D. R. Heath-Brown, *The fourth power moment of the Riemann zeta-function*, Proceedings of the London Mathematical Society, vol. s3-38, no. 3, pp. 385–422, 1979.
- [17] D. R. Heath-Brown, *Fractional moments of the Riemann zeta-function*, Journal of the London Mathematical Society, vol. 24, no. 2, pp. 65–78, 1981.
- [18] D. R. Heath-Brown, *A new k -th derivative estimate for exponential sums via Vinogradov's mean value*. Proceedings of the Steklov Institute of Mathematics, vol. 296, pp. 88–103, 2017.
- [19] M. N. Huxley, *On the difference between consecutive primes*, Inventiones Mathematicae, vol. 15, pp. 164–170, 1972.
- [20] A. E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proceedings of the London Mathematical Society, vol. s2-27, no. 1, pp. 273–300, 1928.
- [21] A. E. Ingham, *On the estimation of $N(\sigma, T)$* , Quarterly Journal of Mathematics, vol. 11, no. 1, pp. 201–202, 1940.
- [22] A. Ivić, *The Riemann zeta-function: theory and applications*, Dover Publications, 2003.
- [23] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society, Colloquium Publications, vol. 53, 2004.
- [24] J. P. Keating and N. C. Snaith, *Random Matrix Theory and $\zeta(1/2 + it)$* , Communications in Mathematical Physics, vol. 214, pp. 57–89, 2000.
- [25] N. M. Korobov, *Estimates of trigonometric sums and their applications*, Uspekhi Matematicheskikh Nauk, vol. 13, pp. 185–192, 1958. (Russian)
- [26] J. E. Littlewood, *On the zeroes of the Riemann zeta-function*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 22, no. 3, pp. 295–318, 1924.
- [27] H. L. Montgomery, *Topics in multiplicative number theory*, Lecture notes in mathematics, vol. 227, Springer-Verlag, 1971.
- [28] N. Ng, *The sixth moment of the Riemann zeta function and ternary additive divisor sums*, Discrete Analysis, vol. 2021, Article no. 6.
- [29] N. Ng, Q. Shen and P.-J. Wong, *The eighth moment of the Riemann zeta function*, Journal of the European Mathematical Society, published online first, DOI: 10.4171/JEMS/1568, 2025.
- [30] M. Radziwiłł, *The 4.36-th moment of the Riemann zeta-function*, International Mathematics Research Notices, vol. 2012, no. 18, pp. 4245–4259, 2012.

- [31] K. Ramachandra, *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series*, Hardy-Ramanujan Journal, vol. 3, pp. 1–15, 1980.
- [32] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Größe*, Monatsberichte der Berliner Akademie, November 1859, pp. 671–680.
- [33] R. Shakarchi and E. M. Stein, *Complex analysis*, Princeton Lectures in Analysis II, Princeton University Press, 2003.
- [34] K. Soundararajan, *Moments of the Riemann zeta-function*, Annals of Mathematics, vol. 170, no. 2, pp. 981–993, 2009.
- [35] T. Tao, T. Trudgian, A. Yang, *New exponent pairs, zero density estimates, and zero additive energy estimates: a systematic approach*, Mathematics of Computation, in press, DOI: <https://doi.org/10.1090/mcom/4138>.
- [36] T. Tao, T. Trudgian and A. Yang, *The analytic number theory exponent database*, <https://github.com/teorth/expdb>, accessed May, 2026.
- [37] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Graduate Studies in Mathematics, vol. 163, American Mathematical Society, 2015.
- [38] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.
- [39] T. S. Trudgian and A. Yang, *Toward optimal exponent pairs*, Mathematics of Computation, vol. 94, pp. 1467–1502, 2025.
- [40] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bulletin of the American Mathematical Society (New Series), vol. 12, no. 2, pp. 183–216, 1985.
- [41] I. M. Vinogradov, *A new estimate for $\zeta(1 + it)$* , Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya, vol. 22, pp. 161–164, 1958. (Russian)
- [42] G. N. Watson and E. T. Whittaker, *A course of modern analysis*, 5th ed., Cambridge University Press, 2021.

DEPARTMENT OF MATHEMATICS
Celestijnenlaan 200B bus 2400
3001 LEUVEN, BELGIUM
tel. + 32 16 32 70 07
fax + 32 16 32 79 98
www.kuleuven.be

